Computing persistent Stiefel-Whitney classes of line bundles

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Abstract. We propose a definition of persistent Stiefel-Whitney classes of vector bundle filtrations. It relies on seeing vector bundles as subsets of some Euclidean spaces. The usual Čech filtration of such a subset can be endowed with a vector bundle structure, that we call a Čech bundle filtration. We show that this construction is stable and consistent. When the dataset is a finite sample of a line bundle, we implement an effective algorithm to compute its persistent Stiefel-Whitney classes. In order to use simplicial approximation techniques in practice, we develop a notion of weak simplicial approximation. As a theoretical example, we give an in-depth study of the normal bundle of the circle, which reduces to understanding the persistent cohomology of the torus knot (1,2).

Numerical experiments. A Python notebook containing illustrations can be found at `https://github.com/raphaeltinarrage/PersistentCharacteristicClasses/blob/master/Demo.ipynb`.

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Contents

1 Introduction

1.1 Statement of the problem

Let $\mathcal{M}_0$ and $\mathcal{M}_0'$ denote the torus and the Klein bottle. Only one of them is orientable, hence these two manifolds are not homeomorphic. Let $\mathbb{Z}_2$ be the field with two elements. Observe that the cohomology groups of $\mathcal{M}_0$ and $\mathcal{M}_0'$ over $\mathbb{Z}_2$ are equal:

$$\begin{aligned} H^0(\mathcal{M}_0) &= H^0(\mathcal{M}_0') = \mathbb{Z}_2, \\ H^1(\mathcal{M}_0) &= H^1(\mathcal{M}_0') = \mathbb{Z}_2 \times \mathbb{Z}_2, \\ H^2(\mathcal{M}_0) &= H^2(\mathcal{M}_0') = \mathbb{Z}_2. \end{aligned}$$

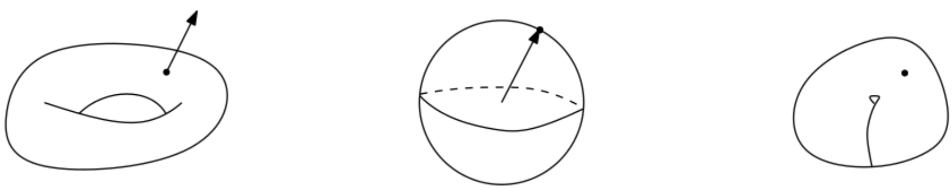
Therefore, the cohomology groups alone do not permit to differenciate the manifolds $\mathcal{M}_0$ and $\mathcal{M}_0'$. To do so, several refinements from algebraic topology may be used. For example, the first cohomology groups $H^1(\mathcal{M}_0)$ and $H^1(\mathcal{M}_0')$, or the second ones $H^2(\mathcal{M}_0)$ and $H^2(\mathcal{M}_0')$ are distinct when computed over the rings $\mathbb{Z}$ or $\mathbb{Z}_p$, $p \geq 3$. Also, the cup product structures on the cohomology rings $H^*(\mathcal{M}_0)$ and $H^*(\mathcal{M}_0')$ are distinct, even over $\mathbb{Z}_2$. In this paper, we will consider another invariant associated to $\mathcal{M}_0$ and $\mathcal{M}_0'$: the characteristic classes of their vector bundles. For instance, if we equip $\mathcal{M}_0$ and $\mathcal{M}_0'$ with their tangent bundles, their first Stiefel-Whitney classes are distinct: only one of them is zero. Hence we are able to differenciate these two manifolds.

$H^*(\mathcal{M}_0) = \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$
$w_1(\tau_{\mathcal{M}_0}) = 0$

$H^*(\mathcal{M}_0') = \mathbb{Z}_2[x,y]/\langle x^3, x^2y^{-2}, xy\rangle$
$w_1(\tau_{\mathcal{M}_0'}) = x$

Figure 1: The cohomology rings of $\mathcal{M}_0$ and $\mathcal{M}_0'$ over $\mathbb{Z}_2$, and the first Stiefel-Whitney classes of their respective tangent bundles $\tau_{\mathcal{M}_0}$ and $\tau_{\mathcal{M}_0'}$.

In general, if X is a topological space endowed with a vector bundle ξ of dimension d, there exists a collection of cohomology classes $w_1(\xi), ..., w_d(\xi)$, the Stiefel-Whitney classes, such that $w_i(\xi)$ is an element of the cohomology group $H^i(\mathcal{M}_0)$ over $\mathbb{Z}_2$ for $i \in [1,d]$. We discuss in Subsection 1.5 the interpretation of these classes. As we explain in Subsection 1.4, defining a vector bundle over a compact space $\mathcal{M}_0$ is equivalent to defining a continuous map $p\colon \mathcal{M}_0 \to \mathcal{G}_d(\mathbb{R}^m)$ for m large enough, where $\mathcal{G}_d(\mathbb{R}^m)$ is the Grassmann manifold of d-planes in $\mathbb{R}^m$. Such a map is called a classifying map for ξ. It is closely related to the Gauss map of submanifolds of $\mathbb{R}^3$.



$$\mathcal{M} \longrightarrow \mathbb{S}_2 \longrightarrow \mathcal{G}_1(\mathbb{R}^3)$$

Figure 2: If $\mathcal{M}$ is an orientable 2-submanifold of $\mathbb{R}^3$, the Gauss map $g\colon \mathcal{M} \to \mathbb{S}_2$ maps every $x \in \mathcal{M}$ to a normal vector of $\mathcal{M}$ at x. By post-composing this map with the usual quotient map $\mathbb{S}_2 \to \mathcal{G}_1(\mathbb{R}^3)$, we obtain a classifying map $f\colon \mathcal{M} \to \mathcal{G}_1(\mathbb{R}^3)$ for the normal bundle of $\mathcal{M}$.

Given a classifying map $p\colon X \to \mathcal{G}_d(\mathbb{R}^m)$ of a vector bundle ξ, the Stiefel-Whitney classes $w_1(\xi), ..., w_d(\xi)$ can be defined by pushing forward some particular classes of the Grassmannian via the induced map in cohomology $p^*\colon H^*(X) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^m))$.

In order to translate these considerations in a persistent-theoretic setting, suppose that we are given a dataset of the form (X, p), where X is a finite subset of $\mathbb{R}^n$, and p is a map $p\colon X \to \mathcal{G}_d(\mathbb{R}^m)$. Denote by $(X^t)_{t\geq 0}$ the Čech filtration of X, which is the collection of the t-thickenings X^t of X in the ambient space $\mathbb{R}^n$. In order to define some persistent Stiefel-Whitney classes, one would try to extend the map $p\colon X \to \mathcal{G}_d(\mathbb{R}^m)$ to $p^t\colon X^t \to \mathcal{G}_d(\mathbb{R}^m)$. However, the author did not find any interesting way to extend this map.

To adopt another point of view, (X, p) can be seen as a subset of $\mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m)$, via $\check{X} = \{(x, p(x)), x \in X\}$. The Grassmann manifold $\mathcal{G}_d(\mathbb{R}^m)$ can be naturally embedded in the matrix space $\mathcal{M}(\mathbb{R}^m)$, hence $\check{X}$ can be seen as a subset $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. If $(\check{X}^t)_{t\geq 0}$ denotes the Čech filtration of $\check{X}$ in the ambient space $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, then a natural map $p^t\colon \check{X}^t \to \mathcal{G}_d(\mathbb{R}^m)$ can be defined: map a point $(x, A) \in \check{X}^t$ to the projection of A on $\mathcal{G}_d(\mathbb{R}^m)$, seen as a subset of $\mathcal{M}(\mathbb{R}^m)$. Using the extended maps $p^t\colon \check{X}^t \to \mathcal{G}_d(\mathbb{R}^m)$, we are able to define a notion of persistent Stiefel-Whitney classes (Definition 2.2). The nullity of a persistent Stiefel-Whitney class is summarized in a diagram that we call a lifebar.

As an example, consider the embedding of the torus $u\colon \mathcal{M}_0 \to \mathcal{M} \subset \mathbb{R}^3$ depicted in Figure 3. Denote P_x the tangent space of $\mathcal{M}$ at x. The set $\check{\mathcal{M}} = \{(x, P_x), x \in \mathcal{M}\}$ can be seen as a subset of $\mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3)$.

Figure 3: The submanifold $\mathcal{M} \subset \mathbb{R}^3$, and the submanifold $\check{\mathcal{M}} \subset \mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3) \simeq \mathbb{R}^{12}$ projected in a 3-dimensional subspace via PCA.

The lifebar of the first persistent Stiefel-Whitney class of this torus is depicted in Figure 4. The bar is hatched, which means that the class is zero all along the filtration. This is coherent with the actual first Stiefel-Whitney class of the normal bundle of the torus, which is zero.

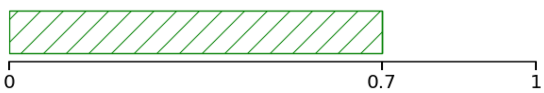


Figure 4: Lifebar of the first persistent Stiefel-Whitney class of $\check{\mathcal{M}}$. It is only defined on the interval $\left[0, \frac{\sqrt{2}}{2}\right)$ (see Definition 2.3).

To continue, consider the immersion of the Klein bottle $u'\colon \mathcal{M}_0' \to \mathcal{M}' \subset \mathbb{R}^3$ depicted in Figure 5. For $x_0 \in \mathcal{M}_0'$, denote P_{x_0} the tangent space of $\mathcal{M}_0'$ at x_0, seen in $\mathbb{R}^3$. The set $\check{\mathcal{M}}' = \{(u(x_0), P_{x_0}), x_0 \in \mathcal{M}_0'\}$ can be seen as a subset of $\mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3)$. Note that $\check{\mathcal{M}}'$ is a submanifold (diffeomorphic to the Klein bottle), while $\mathcal{M}'$ is not.

Figure 5: The set $\mathcal{M}' \subset \mathbb{R}^3$, and the submanifold $\check{\mathcal{M}}' \subset \mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3) \simeq \mathbb{R}^{12}$ projected in a 3-dimensional subspace via PCA.

Just as before, we can define persistent Stiefel-Whitney classes over the Čech filtration of $\check{\mathcal{M}}'$. Figure 6 represents the lifebar of the first Stiefel-Whitney class of this filtration. The bar is filled, which means that the class is nonzero all along the filtration. This is coherent with the first Stiefel-Whitney class of the normal bundle of the Klein bottle, which is nonzero.

Figure 6: Lifebar of the first persistent Stiefel-Whitney class of $\check{\mathcal{M}}'$.

The construction we propose is defined for any subset $X \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. In particular, it can be applied to finite samples of $\check{\mathcal{M}}$ and $\check{\mathcal{M}}'$. We prove that it is stable and consistent (Theorems 2.3 and 2.6). As an illustration, Figure 7 represents the lifebars of the first persistent Stiefel-Whitney classes of samples X and X' of $\check{\mathcal{M}}$ and $\check{\mathcal{M}}'$. Observe that they are close to the original ones.

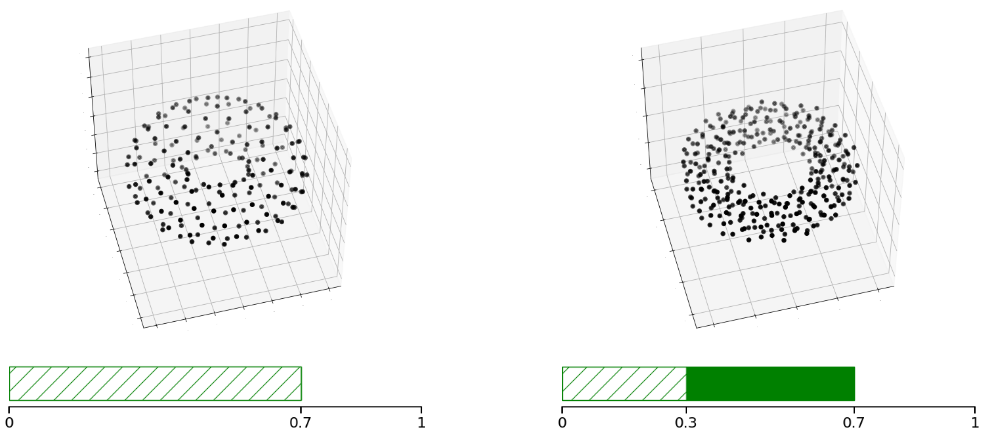


Figure 7: Left: a sample of $\check{\mathcal{M}} \subset \mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3)$, seen in $\mathbb{R}^3$, and the lifebar of its first persistent Stiefel-Whitney class. Right: same for $\check{\mathcal{M}}'$.

1.2 Notations

We adopt the following notations:

- I denotes a set, $\mathrm{card}(I)$ its cardinal and I^c its complement.
- $\mathbb{R}^n$ and $\mathbb{R}^m$ denotes the Euclidean spaces of dimension n and m, E denotes a Euclidean space.

- $\mathcal{M}(\mathbb{R}^m)$ the vector space of $m \times m$ matrices, $\mathcal{G}_d(\mathbb{R}^m)$ the Grassmannian of d-subspaces of $\mathbb{R}^m$, and $\mathbb{S}_k \subset \mathbb{R}^{k+1}$ the unit k-sphere.
- $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^n$, $\|\cdot\|_{\mathrm{F}}$ the Frobenius norm on $\mathcal{M}(\mathbb{R}^m)$, $\|\cdot\|_\gamma$ the norm on $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ defined as $\|(x,A)\|_\gamma^2 = \|x\|^2 + \gamma^2\|A\|_{\mathrm{F}}^2$ where $\gamma > 0$ is a parameter.
- $\mathbb{X} = (X^t)_{t\in T}$ denotes a set filtration. $\mathbb{V}[\mathbb{X}]$ denotes the corresponding persistent cohomology module. If X is a subset of E, then $\mathbb{X} = (X^t)_{t\in T}$ denotes the Čech set filtration of X.
- $(\mathbb{V}, \mathbb{v})$ denotes a persistence module over T, with $\mathbb{V} = (V^t)_{t\in T}$ a family of vector spaces, and $\mathbb{v} = (v_s^t \colon X^s \leftarrow X^t)_{s\leq t\in T}$ a family of linear maps.
- $\mathcal{U}$ denotes a cover of a topological space, and $\mathcal{N}(\mathcal{U})$ its nerve. $\mathbb{S} = (S^t)_{t\in T}$ denotes a simplicial filtration.
- $(\mathbb{X}, \mathbb{p})$ denotes a vector bundle filtration, with $\mathbb{X}$ a set filtration, and $\mathbb{p} = (p^t)_{t\in T}$ a family of maps $p^t \colon X^t \to \mathcal{G}_d(\mathbb{R}^m)$. If X is a subset of $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, then $(\mathbb{X}, \mathbb{p})$ denotes the Čech bundle filtration associated to X.
- If X is a topological space, $H^*(X)$ denotes its cohomology ring, and $H^i(X)$ the ith cohomology group. If $f\colon X \to Y$ is a continuous map, $f^*\colon H^*(X) \leftarrow H^*(Y)$ is the map induced in cohomology.
- If ξ is a vector bundle, $w_i(\xi)$ denotes its ith Stiefel-Whitney class. If $(\mathbb{X}, \mathbb{p})$ is a vector bundle filtration, $w_i(\mathbb{p})$ denotes the ith persistent Stiefel-Whitney class, with $w_i(\mathbb{p}) = (w_i^t(\mathbb{p}))_{t\in T}$ (see Definition 2.2).
- If A is a subset of E, then $\mathrm{med}\,(A)$ denotes its medial axis, $\mathrm{reach}(A)$ its reach, $\mathrm{dist}\,(\cdot, A)$ the distance to A (see Subsection 1.3). The projection on A is denoted $\mathrm{proj}\,(\cdot, A)$ or $\mathrm{proj}_A\,(\cdot)$. $d_{\mathrm{H}}(\cdot,\cdot)$ denotes the Hausdorff distance between two sets of E.
- If K is a simplicial complex, K^i denotes its i-skeleton. For every vertex $v \in K^0$, $\mathrm{St}\,(v)$ and $\overline{\mathrm{St}}\,(v)$ denote its open and closed star. The topological realization of K is denoted $|K|$, and the topological realization of a simplex $\sigma \in K$ is $|\sigma|$. The face map is denoted $\mathcal{F}_K \colon |K| \to K$ (see Subsection 3.1).
- If $f\colon K \to L$ is a simplicial map, $|f|\colon |K| \to |L|$ denotes its topological realization. The ith barycentric subdivision of the simplicial complex K is denoted $\mathrm{sub}^i\,(K)$ (see Subsection 3.2).

1.3 Background on persistent cohomology

In the following, we consider interleavings of filtrations, interleavings of persistence modules and their associated pseudo-distances. Their definitions, in the context of cohomology, are recalled in this subsection. Compared to the standard definitions of persistent homology, the arrows go backward. Let $T \subseteq [0, +\infty)$ be an interval that contains 0, and let E be a Euclidean space.

Filtrations of sets and simplicial complexes. A family of subsets $\mathbb{X} = (X^t)_{t\in T}$ of E is a *filtration* if it is non-decreasing for the inclusion, i.e. for any $s, t \in T$, if $s \leq t$ then $X^s \subseteq X^t$. Given $\epsilon \geq 0$, two filtrations $\mathbb{X} = (X^t)_{t\in T}$ and $\mathbb{Y} = (Y^t)_{t\in T}$ of E are *ϵ-interleaved* if, for every $t \in T$, $X^t \subseteq Y^{t+\epsilon}$ and $Y^t \subseteq X^{t+\epsilon}$. The interleaving pseudo-distance between $\mathbb{X}$ and $\mathbb{Y}$ is defined as the infimum of such ϵ:

$$d_i(\mathbb{X}, \mathbb{Y}) = \inf\{\epsilon,\ \mathbb{X} \text{ and } \mathbb{Y} \text{ are } \epsilon\text{-interleaved}\}.$$

Filtrations of simplicial complexes and their interleaving distance are similarly defined: given an abstract simplex S, a *filtration of* S is a non-decreasing family $\mathbb{S} = (S^t)_{t\in T}$ of subcomplexes of S. The interleaving pseudo-distance between two filtrations $(S_1^t)_{t\in T}$ and $(S_2^t)_{t\in T}$ of S is the infimum of the $\epsilon \geq 0$ such that they are ϵ-interleaved, i.e., for any $t \in T$, we have $S_1^t \subseteq S_2^{t+\epsilon}$ and $S_2^t \subseteq S_1^{t+\epsilon}$.

Persistence modules and interleavings. Let k be a field. A *persistence module* over T is a pair $(\mathbb{V}, \mathbb{v})$ where $\mathbb{V} = (V^t)_{t \in T}$ is a family of k-vector spaces, and $\mathbb{v} = (v_s^t)_{s \leq t \in T}$ is a family of linear maps $v_s^t \colon V^s \leftarrow V^t$ such that:

- for every $t \in T$, $v_t^t \colon V^t \leftarrow V^t$ is the identity map,
- for every $r, s, t \in T$ such that $r \leq s \leq t$, $v_r^s \circ v_s^t = v_r^t$.

When there is no risk of confusion, we may denote a persistence module by $\mathbb{V}$ instead of $(\mathbb{V}, \mathbb{v})$. Given $\epsilon \geq 0$, an *ϵ-morphism* between two persistence modules $(\mathbb{V}, \mathbb{v})$ and $(\mathbb{W}, \mathbb{w})$ is a family of linear maps $(\phi_t \colon V^t \to W^{t-\epsilon})_{t \geq \epsilon}$ such that the following diagrams commute for every $\epsilon \leq s \leq t$:

$$\begin{tikzcd}
V^s \arrow[d, "\phi_s"'] & V^t \arrow[l, "v_s^t"] \arrow[d, "\phi_t"] \\
W^{s-\epsilon} & W^{t-\epsilon} \arrow[l, "w_{s-\epsilon}^{t-\epsilon}"]
\end{tikzcd}$$

If $\epsilon = 0$ and each ϕ_t is an isomorphism, the family $(\phi_t)_{t \in T}$ is an *isomorphism* of persistence modules. An *ϵ-interleaving* between two persistence modules $(\mathbb{V}, \mathbb{v})$ and $(\mathbb{W}, \mathbb{w})$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t-\epsilon})_{t \geq \epsilon}$ and $(\psi_t \colon W^t \to V^{t-\epsilon})_{t \geq \epsilon}$ such that the following diagrams commute for every $t \geq 2\epsilon$:

$$\begin{tikzcd}
V^{t-2\epsilon} & & V^t \arrow[ll, "v_{t-2\epsilon}^t"'] \arrow[dl, "\phi_t"] \\
& W^{t-\epsilon} \arrow[ul, "\psi_{t-\epsilon}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{t-\epsilon} \arrow[dl, "\phi_{t-\epsilon}"'] & \\
W^{t-2\epsilon} & & W^t \arrow[ll, "w_{t-2\epsilon}^t"] \arrow[ul, "\psi_t"']
\end{tikzcd}$$

The interleaving pseudo-distance between $(\mathbb{V}, \mathbb{v})$ and $(\mathbb{W}, \mathbb{w})$ is defined as

$$d_i(\mathbb{V}, \mathbb{W}) = \inf\{\epsilon \geq 0,\ \mathbb{V} \text{ and } \mathbb{W} \text{ are } \epsilon\text{-interleaved}\}.$$

In some cases, the proximity between persistence modules is expressed with a function. Let $T' \subseteq T$ and $\eta \colon T' \to T$ be a non-increasing function such that for any $t \in T'$, $\eta(t) \leq t$. A η-interleaving between two persistence modules $(\mathbb{V}, \mathbb{v})$ and $(\mathbb{W}, \mathbb{w})$ is a pair of families of linear maps $(\phi_t \colon V^t \to W^{\eta(t)})_{t \in T'}$ and $(\psi_t \colon W^t \to V^{\eta(t)})_{t \in T'}$ such that the following diagrams commute for every $t \in T'$:

$$\begin{tikzcd}
V^{\eta(\eta(t))} & & V^t \arrow[ll, "v_{\eta(\eta(t))}^t"'] \arrow[dl, "\phi_t"] \\
& W^{\eta(t)} \arrow[ul, "\psi_{\eta(t)}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{\eta(t)} \arrow[dl, "\phi_{\eta(t)}"'] & \\
W^{\eta(\eta(t))} & & W^t \arrow[ll, "w_{\eta(\eta(t))}^t"] \arrow[ul, "\psi_t"']
\end{tikzcd}$$

When η is $t \mapsto t - c$ for some $c > 0$, it is called an *additive c-interleaving* and corresponds with the previous definition. When η is $t \mapsto ct$ for some $0 < c < 1$, it is called a *multiplicative c-interleaving.*

Persistence diagrams. A persistence module $(\mathbb{V}, \mathbb{v})$ is said to be *pointwise finite-dimensional* if for every $t \in T$, V^t is finite-dimensional. This implies that we can define a notion of persistence diagram [BCB18, Theorem 1.2]. It is based on the algebraic decomposition of the persistence module into interval modules. Moreover, given two pointwise finite-dimensional persistence modules $\mathbb{V}, \mathbb{W}$ with persistence diagrams $D(\mathbb{V}), D(\mathbb{W})$, the so-called isometry theorem states that $d_b(D(\mathbb{V}), D(\mathbb{W})) = d_i(\mathbb{V}, \mathbb{W})$ where $d_i(\cdot, \cdot)$ denotes the interleaving distance, and $d_b(\cdot, \cdot)$ denotes the bottleneck distance between diagrams.

More generally, the persistence module $(\mathbb{V}, \mathbb{v})$ is said to be *q-tame* if for every $s, t \in T$ such that $s < t$, the map v_s^t is of finite rank. The q-tameness of a persistence module ensures that we can still define a notion of persistence diagram, even though the module may not be decomposable into interval modules. Moreover, the isometry theorem still holds [CdSGO16, Theorem 4.11].

Relation between filtrations and persistence modules. Applying the singular cohomology functor to a set filtration gives rise to a persistence module whose linear maps between cohomology groups are induced by the inclusion maps between sets. As a consequence, if two filtrations are ϵ-interleaved, then their associated cohomology persistence modules are also ϵ-interleaved, the interleaving homomorphisms being induced by the interleaving inclusion maps. As a consequence of the isometry theorem, if the modules are q-tame, then the bottleneck distance between their persistence diagrams is upperbounded by ϵ.

The same remarks hold when applying the simplicial cohomology functor to simplicial filtrations.

Reach of subsets of E. Let X be any subset of E. Following [Fed59, Definition 4.1], the function *distance to* X is the map $\mathrm{dist}\,(\cdot, X) : y \in E \mapsto \inf\{\|y-x\|, x \in X\}$. A projection of y on X is a point $x \in X$ which attains this infimum. The medial axis of X is the subset $\mathrm{med}\,(X) \subset E$ which consists of points $y \in E$ that admits at least two projections:

$$\mathrm{med}\,(X) = \{y \in E, \exists x, x' \in X, x \neq x', \|y-x\| = \|y-x\| = \mathrm{dist}\,(y, X)\} .$$

The *reach* of X is

$$\mathrm{reach}(X) = \inf\{\|x-y\|, x \in X, y \in \mathrm{med}\,(X)\} .$$

Alternatively, let X^t denote the t-thickening of X, i.e. the subset of points of E at distance at most t from X. Then the reach of X can be defined as the supremum of $t \geq 0$ such that X^t does not intersect $\mathrm{med}\,(X)$.

Suppose that X is closed and let $\mathrm{reach}(X)$ be the reach of X. One shows that each X^t deform retracts onto X for $0 \leq t < \mathrm{reach}(X)$. Besides, if Y is any other subset of E with Hausdorff distance $d_{\mathrm{H}}(X, Y) \leq \epsilon$, then for any $t \in [4\epsilon, \mathrm{reach}(X) - 3\epsilon)$, Y^t deform retracts on X [CCSL09, Theorem 4.6, case $\mu = 1$].

Weak feature size of compact subsets of E. Let X be any compact subset of E, and denote by d_X the distance function to X. It is not differentiable in general. However, one can define a generalized gradient vector field $\nabla_X : E \to E$, as in [BCY18, Section 9.2]. A point $x \in E$ is called a critical point of d_X if $\nabla_X(x) = 0$. One shows that x is a critical point of d_X if it lies in the convex hull of its projections on X. The *weak feature size* of X is defined as

$$\mathrm{wfs}\,(X) = \inf\{\mathrm{dist}\,(x, X)\,, x \text{ is a critical point of } d_X\} .$$

The Isotopy Lemma [BCY18, Theorem 9.5] states that for every $s, t \in \mathbb{R}$ such that $0 < s \leq t < \mathrm{wfs}\,(X)$, the thickening X^t is isotopic to X^s. This isotopy can be chosen to be a deformation retraction. If X admits a positive reach, we deduce that the thickenings X^t deform retracts on X. The weak feature size and reach of X satisfy the inequality $\mathrm{reach}(X) \leq \mathrm{wfs}\,(X)$.

Čech set filtrations. Let X denote any subset of E. The *Čech set filtration* associated to X is the filtration of E defined as the collection of subsets $\mathbb{X} = (X^t)_{t\geq 0}$, where X^t denotes the t-thickening of X in E, that is, $X^t = \{x \in E, \mathrm{dist}\,(x, X) \leq t\}$.

If X is a compact submanifold, then according to the previous considerations about the reach, for every $t \in [0, \mathrm{reach}(X))$, X^t deform retracts on X. Therefore, the corresponding cohomology persistence module is constant on the interval $[0, \tau)$, and is equal to the cohomology of X. Moreover, if Y is any other subset of E with Hausdorff distance $d_{\mathrm{H}}(X, Y) \leq \epsilon$, then the cohomology persistence module of the Čech filtration associated to Y is constant on the interval $[4\epsilon, \tau - 3\epsilon)$ and is equal to the cohomology of X.

Čech simplicial filtrations. Let X denote a finite subset of E and $\mathbb{X} = (X^t)_{t\geq 0}$ its associated Čech set filtration. For all $t \geq 0$, X^t is a union of closed balls of radius t: $X^t = \bigcup_{x\in X} \overline{\mathcal{B}}\,(x, t)$. Consider the simplicial filtration $\mathbb{S} = (S^t)$, where S^t is the nerve of the cover $\mathcal{U}^t$ defined as $\mathcal{U}^t =$

$\{\overline{\mathcal{B}}\,(x,t)\,, x \in X\}$. It is called the *Čech simplicial filtration* associated to X. The persistent nerve lemma [CO08, Lemma 3.4] states that the persistence (singular) cohomology module associated to $\mathbb{X}$ and the persistent (simplicial) cohomology module associated to $\mathbb{S}$ are isomorphic.

1.4 Background on vector bundles

This subsection and the next one follow the presentation of [MS16].

Vector bundles. Let X be a topological space. A *vector bundle* ξ of dimension d consists of a topological space $A = A(\xi)$, the *total space*, a continuous map $\pi = \pi(\xi) \colon A \to X$, the *projection map*, and for every $x \in X$, a structure of d-dimensional vector space on $\pi^{-1}(\{x\})$. Moreover, ξ must satisfies the local triviality condition: for every $x \in X$, there exists a neighborhood $U \subseteq X$ of x and a homeomorphism $h \colon U \times \mathbb{R}^d \to \pi^{-1}(U)$ such that for every $y \in U$, the map $z \mapsto h(y,z)$ defines an isomorphism between the vector spaces $\mathbb{R}^d$ and $\pi^{-1}(\{y\})$.

$$\begin{tikzcd} A(\xi) \arrow[d, "\pi"] \\ X \end{tikzcd} \qquad\qquad \begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi"'] & U \times \mathbb{R}^d \arrow[l, "h"'] \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

In this subsection, the fibers $\pi^{-1}(\{x\})$ will be denoted $F_x(\xi)$.

Isomorphisms of vector bundles. An *isomorphism of vector bundles* ξ, η with common base space X is a homeomorphism $f \colon A(\xi) \to A(\eta)$ which sends each fiber $F_x(\xi)$ isomorphically into $F_{f(x)}(\eta)$. We obtain a commutative diagram

$$\begin{tikzcd} A(\xi) \arrow[rr, "f"] \arrow[dr, "\pi(\xi)"'] & & A(\eta) \arrow[dl, "\pi(\eta)"] \\ & X & \end{tikzcd}$$

The *trivial bundle* of dimension d over X, denoted $\epsilon = \epsilon_X^d$, is defined with the total space $A(\epsilon) = X \times \mathbb{R}^d$, with the projection map π being the projection on the first coordinate, and where each fiber is endowed with the usual vector space structure of $\mathbb{R}^d$. A vector bundle ξ over X is said trivial if it is isomorphic to ϵ.

Operations on vector bundles. If ξ, η are two vector bundles on X, we define their *Whitney sum* $\xi \oplus \eta$ by

$$A(\xi \oplus \eta) = \{(x,a,b), x \in X, a \in F_x(\xi), b \in F_x(\eta)\},$$

where the projection map is given by the projection on the first coordinate, and where the vector space structures are the product structures. If η is a vector bundle on Y and $g \colon X \to Y$ a continuous map, the *pullback bundle* $g^*\xi$ is the vector bundle on X defined by

$$A(g^*\xi) = \{(x,a), x \in X, a \in F_{g(x)}(\xi)\},$$

where the projection map is given by the projection on the first coordinate.

Bundle maps. A *bundle map* between two vector bundles ξ, η with base spaces X and Y is a contiuous map $f \colon A(\xi) \to A(\eta)$ which sends each fiber $F_x(\xi)$ isomorphically into another fiber $F_{x'}(\eta)$. If such a map exists, there exist a unique map $\overline{f}$ which makes the following diagram commute:

$$\begin{array}{ccc} A(\xi) & \xrightarrow{f} & A(\eta) \\ {\scriptstyle \pi(\xi)}\downarrow & & \downarrow{\scriptstyle \pi(\eta)} \\ X & \xrightarrow[\overline{f}]{} & Y \end{array}$$

In this case, ξ is isomorphic to the pullback bundle $\overline{f}^*\eta$ [MS16, Lemma 3.1]. We say that the map $\overline{f}$ *covers* f.

Universal bundles. Let $0 < d \leq m$. The Grassmann manifold $\mathcal{G}_d(\mathbb{R}^m)$ is a set which consists of all d-dimensional linear subspaces of $\mathbb{R}^m$. It can be given a smooth manifold structure. When $d = 1$, $\mathcal{G}_1(\mathbb{R}^m)$ corresponds to the real projective space $\mathbb{P}_n(\mathbb{R})$. On $\mathcal{G}_d(\mathbb{R}^m)$, there exists a canonical vector bundle of dimension d, denoted γ_d^m. It consists in the total space

$$A(\gamma_d^m) = \{(V, v), V \in \mathcal{G}_d(\mathbb{R}^m), v \in V\} \subset \mathcal{G}_d(\mathbb{R}^m) \times \mathbb{R}^m,$$

with the projection map on the first coordinate, and the linear structure inherited from $\mathbb{R}^m$.

Lemma 1.1 ([MS16, Lemma 5.3])**.** *Let ξ be vector bundle of dimension d over a compact space X. Then for m large enough, there exists a bundle map from ξ to γ_d^m.*

If such a bundle map $f : \xi \to \gamma_d^m$ exists, then ξ is isomorphic to the pullback $\overline{f}^*\gamma_d^m$, where $\overline{f}$ denotes the map that f covers.

In order to avoid mentionning m, it is convenient to consider the infinite Grassmannian. The infinite Grassmann manifold $\mathcal{G}_d(\mathbb{R}^\infty)$ is the set of all d-dimensional linear subspaces of $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$ is the vector space of series with a finite number of nonzero terms. The infinite Grassmannian is topologized as the direct limit of the sequence $\mathcal{G}_d(\mathbb{R}^d) \subset \mathcal{G}_d(\mathbb{R}^{d+1}) \subset \mathcal{G}_d(\mathbb{R}^{d+2}) \subset \cdots$. Just as before, there exists on $\mathcal{G}_d(\mathbb{R}^\infty)$ a canonical bundle γ_d^∞. It is called a *universal bundle*, for the following reason:

Lemma 1.2 ([MS16, Lemma 5.3])**.** *if ξ is vector bundle of dimension d over a paracompact space X, then there exists a bundle map from $\xi \to \gamma_d^\infty$.*

Such a bundle map is denoted $f_\xi : A(\xi) \to A(\gamma_d^\infty)$. The underlying map between base spaces, denoted $\overline{f}_\xi : X \to \mathcal{G}_d(\mathbb{R}^\infty)$, is called a *classifying map for ξ*. As before, ξ is isomorphic to the pullback $(\overline{f}_\xi)^*\gamma_d^\infty$. Note that if f is a bundle map given by Lemma 1.1, then the following composition is a classifying map for ξ:

$$X \xrightarrow{\overline{f}} \mathcal{G}_d(\mathbb{R}^m) \hookrightarrow \mathcal{G}_d(\mathbb{R}^\infty).$$

A correspondance. Let ξ, η be bundles over X, and let $\overline{f_\xi}, \overline{f_\eta}$ be classifying maps. If these maps are homotopic, one shows that the bundles ξ and η are isomorphic. The following theorem states that the converse is also true.

Theorem 1.3 ([MS16, Corollary 5.10])**.** *Let X be a paracompact space. There exists a bijection between the vector bundles over X (up to isomorphism) and the continuous maps $X \to \mathcal{G}_d(\mathbb{R}^\infty)$ (up to homotopy). It is given by $\xi \mapsto \overline{f}_\xi$, where $\overline{f}_\xi$ denotes a the classifying map for ξ.*

This result leads to the following convention:

In the rest of this paper, we will consider that vector bundles are given as a continuous maps $X \to \mathcal{G}_d(\mathbb{R}^m)$ or $X \to \mathcal{G}_d(\mathbb{R}^\infty)$.

1.5 Background on Stiefel-Whitney classes

The Stiefel-Whitney classes are a particular instance of the theory of characteristic classes, with coefficient group being $\mathbb{Z}_2$. We first define them axiomatically, and then describe their construction.

Axioms for Stiefel-Whitney classes. To each vector bundle ξ over a paracompact base space X, one associates a sequence of cohomology classes

$$w_i(\xi) \in H^i(X, \mathbb{Z}_2), \qquad i \in \mathbb{N},$$

called the *Stiefel-Whitney classes of* ξ. These classes satisfy:

- **Axiom 1:** w_0 is equal to $1 \in H^0(X, \mathbb{Z}_2)$, and if ξ is of dimension d then $w_i(\xi) = 0$ for $i > d$.
- **Axiom 2:** if $f: \xi \to \eta$ is a bundle map, then $w_i(\xi) = \overline{f}^* w_i(\eta)$, where $\overline{f}^*$ is the map in cohomology induced by the underlying map $\overline{f}$.
- **Axiom 3:** if ξ, η are bundles over the same base space X, then for all $k \in \mathbb{N}$, $w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta)$, where $\smile$ denotes the cup product.
- **Axiom 4:** $w_1(\gamma_1^1) \neq 0$, where γ_1^1 denotes the universal bundle of the projective line $\mathcal{G}_1(\mathbb{R}^2)$.

The Stiefel-Whitney classes are invariants of vector bundles, and carry topological information. For instance, the following lemma shows that the first Stefel-Whitney class detects orientability.

Proposition 1.4 ([MS16, Lemma 11.6 and Problem 12-A])**.** *If X is a compact manifold and τ its tangent bundle, then X is orientable if and only if $w_1(\tau) = 0$.*

Construction of the Stiefel-Whitney classes. The cohomology ring of the Grassmann manifolds admits a simple description: $H^*(G_d(\mathbb{R}^\infty), \mathbb{Z}_2)$ is the free abelian ring generated by d elements $w_1, ..., w_d$. As a graded algebra, the degree of these elements are $|w_1| = 1, ..., |w_d| = d$ [MS16, Theorem 7.1]. Hence we can write

$$H^*(G_d(\mathbb{R}^\infty), \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, ..., w_d].$$

In particular, the infinite projective space $\mathbb{P}_\infty = G_1(\mathbb{R}^\infty)$ space has cohomology $H^*(\mathbb{P}_\infty, \mathbb{Z}_2) = \mathbb{Z}_2[w_1]$, the polynomial ring.

The generators $w_1, ..., w_d$ can be seen as the Stiefel-Whitney classes of the universal bundle γ_d^∞ on $\mathcal{G}_d(\mathbb{R}^\infty)$. Now, for any vector bundle ξ, define

$$w_i(\xi) = \overline{f}_\xi^*(w_i),$$

where $\overline{f}_\xi: X \to \mathcal{G}_d(\mathbb{R}^\infty)$ is a classifying map for ξ (as in Theorem 1.3) and $\overline{f}_\xi^*: H^*(X) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^\infty))$ the induced map in cohomology. This construction yields the Stiefel-Whitney classes:

Theorem 1.5 ([MS16, Theorem 7.3])**.** *Defined this way, the classes satisfy the four axioms. And they are unique.*

2 Persistent Stiefel-Whitney classes

2.1 Definition

Let E be a Euclidean space, and $\mathbb{X} = (X^t)_{t\in T}$ a set filtration of E (see Subsection 1.3). Let us denote by i_s^t the inclusion map from X^s to X^t. In order to define persistent Stiefel-Whitney classes, we have to give such a filtration a vector bundle structure.

Definition 2.1 (Vector bundle filtrations)**.** A vector bundle filtration of dimension d on E is a couple $(\mathbb{X}, \mathbb{p})$ where $\mathbb{X} = (X^t)_{t\in T}$ is a set filtration of E and $\mathbb{p} = (p^t)_{t\in T}$ a family of continuous maps $p^t: X^t \to \mathcal{G}_d(\mathbb{R}^\infty)$ such that, for every $s, t \in T$ with $s \leq t$, we have $p^t \circ i_s^t = p^s$. In other words, the following diagram commutes:

$$\begin{tikzcd}
X^s \arrow[rr, "i_s^t"] \arrow[dr, "p^s"'] & & X^t \arrow[dl, "p^t"] \\
& \mathcal{G}_d(\mathbb{R}^\infty) &
\end{tikzcd}$$

Let us fix a $t \in T$. The map $p^t\colon X^t \to \mathcal{G}_d(\mathbb{R}^\infty)$ gives the topological space X^t a vector bundle structure, as discussed in Subsection 1.4. Following Subsection 1.5, the induced map in cohomology, $(p^t)^*$, allows to define the Stiefel-Whitney classes of this vector bundle. Let us introduce some notations. The Stiefel-Whitney classes of $\mathcal{G}_d(\mathbb{R}^\infty)$ are denoted $w_1, ..., w_d$. The Stiefel-Whitney classes of the vector bundle (X^t, p^t) are denoted $w_1^t(\mathbb{p}), ..., w_d^t(\mathbb{p})$, and can be defined as $w_i^t(\mathbb{p}) = (p^t)^*(w_i)$ (as in Theorem 1.5).

$$\begin{array}{rcl}
(p^t)^*\colon H^*\big(X^t\big) & \longleftarrow & H^*\big(\mathcal{G}_d(\mathbb{R}^\infty)\big) \\
w_1^t(\mathbb{p}) & \longleftarrow\!\shortmid & w_1 \\
& \vdots & \\
w_d^t(\mathbb{p}) & \longleftarrow\!\shortmid & w_d
\end{array}$$

Let $(\mathbb{V}, \mathbb{v})$ denote the persistence module corresponding to the filtration $\mathbb{X}$, with $\mathbb{V} = (V^t)_{t\in T}$ and $\mathbb{v} = (v_s^t)_{s\leq t\in T}$. For every $t \in T$, the classes $w_1^t(\mathbb{p}), \cdots, w_d^t(\mathbb{p})$ belong to the vector space V^t. The persistent Stiefel-Whitney classes are defined to be the collection of such classes over t.

Definition 2.2 (Persistent Stiefel-Whitney classes)**.** Let $(\mathbb{X}, \mathbb{p})$ be a vector bundle filtration. The persistent Stiefel-Whitney classes of $(\mathbb{X}, \mathbb{p})$ are the families of classes

$$\begin{aligned}
w_1(\mathbb{p}) &= \big(w_1^t(\mathbb{p})\big)_{t\in T} \\
&\vdots \\
w_d(\mathbb{p}) &= \big(w_d^t(\mathbb{p})\big)_{t\in T}.
\end{aligned}$$

Let $i \in [1, d]$, and consider a persistent Stiefel-Whitney class $w_i(\mathbb{p})$. Note that it satisfies the following property: for all $s, t \in T$ such that $s \leq t$, we have $w_i^s(\mathbb{p}) = v_s^t\big(w_i^t(\mathbb{p})\big)$. As a consequence, if a class $w_i^t(\mathbb{p})$ is given for a $t \in T$, one obtains all the others $w_i^s(\mathbb{p})$, with $s \leq t$, by applying the maps v_s^t. In particular, if $w_i^t(\mathbb{p}) = 0$, then $w_i^s(\mathbb{p}) = 0$ for all $s \in T$ such that $s \leq t$.

Lifebar. In order to visualize the evolution of a persistent Stiefel-Whitney class through the persistence module $(\mathbb{V}, \mathbb{v})$, we propose the following bar representation: the lifebar of $w_i(\mathbb{p})$ is the set

$$\left\{t \in T, w_i^t(\mathbb{p}) \neq 0\right\}.$$

According to the last paragraph, the lifebar of a persistent class is an interval of T, of the form $[t^\dagger, \sup(T))$ or $(t^\dagger, \sup(T))$, where

$$t^\dagger = \inf\left\{t \in T, w_i^t(\mathbb{p}) \neq 0\right\},$$

with the convention $\inf(\emptyset) = \inf(T)$. In order to distinguish the lifebar of a persistent Stiefel-Whitney class from the bars of the persistence barcodes, we draw the rest of the interval hatched.

Figure 8: Example of a lifebar of a persistent Stiefel-Whitney class with $t^\dagger = 0{,}2$ and $\max(T) = 1$.

2.2 Čech bundle filtration

In this subsection, we propose a particular construction of vector bundle filtration, called the Čech bundle filtration. We will work in the ambient space $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Let $\|\cdot\|$ be the usual Euclidean norm on the space $\mathbb{R}^n$, and $\|\cdot\|_{\mathrm{F}}$ the Frobenius norm on the matrix space $\mathcal{M}(\mathbb{R}^m)$. Let $\gamma > 0$. We endow the vector space E with the Euclidean norm $\|\cdot\|_\gamma$ defined for every $(x, A) \in E$ as

$$\|(x,A)\|_\gamma^2 = \|x\|^2 + \gamma^2 \|A\|_{\mathrm{F}}^2. \tag{1}$$

See Subsection 4.4 for a discussion about the parameter γ.

In order to define the Čech bundle filtration, we will first study the usual embedding of the Grassmann manifold $\mathcal{G}_d(\mathbb{R}^m)$ into the matrix space $\mathcal{M}(\mathbb{R}^m)$.

Embedding of $\mathcal{G}_d(\mathbb{R}^m)$. We embed the Grassmannian $\mathcal{G}_d(\mathbb{R}^m)$ into $\mathcal{M}(\mathbb{R}^m)$ via the application which sends a d-dimensional subspace $T \subset \mathbb{R}^m$ to its orthogonal projection matrix P_T. We can now see $\mathcal{G}_d(\mathbb{R}^m)$ as a submanifold of $\mathcal{M}(\mathbb{R}^m)$. Recall that $\mathcal{M}(\mathbb{R}^m)$ is endowed with the Frobenius norm. According to this metric, $\mathcal{G}_d(\mathbb{R}^m)$ is included in the sphere of center 0 and radius $\sqrt{d}$ of $\mathcal{M}(\mathbb{R}^m)$.

In the metric space $(\mathcal{M}(\mathbb{R}^m), \|\cdot\|_{\mathrm{F}})$, consider the distance function to $\mathcal{G}_d(\mathbb{R}^m)$, denoted $\mathrm{dist}\,(\cdot, \mathcal{G}_d(\mathbb{R}^m))$. Let $\mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m))$ denote the medial axis of $\mathcal{G}_d(\mathbb{R}^m)$. It consists in the points $A \in \mathcal{M}(\mathbb{R}^m)$ which admit at least two projections on $\mathcal{G}_d(\mathbb{R}^m)$:

$$\begin{aligned}\mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m)) = \{A \in \mathcal{M}(\mathbb{R}^m), \exists P, P' \in \mathcal{G}_d(\mathbb{R}^m), P \neq P', \\ \|A - P\|_{\mathrm{F}} = \|A - P\|_{\mathrm{F}} = \mathrm{dist}\,(A, \mathcal{G}_d(\mathbb{R}^m))\}.\end{aligned}$$

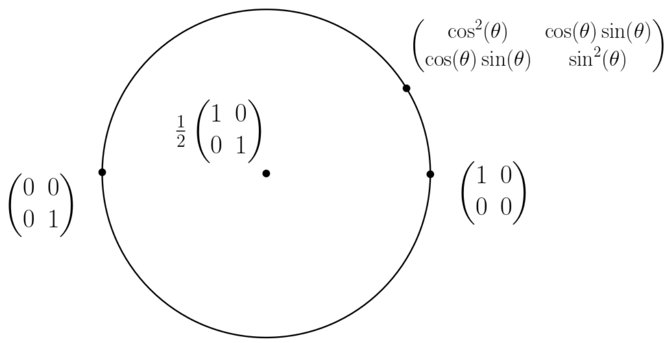


Figure 9: Representation of the Grassmannian $\mathcal{G}_1(\mathbb{R}^2) \subset \mathcal{M}(\mathbb{R}^2) \simeq \mathbb{R}^4$. It is equal to the circle of radius $\frac{\sqrt{2}}{2}$, in the 2-dimensionnal affine space generated by $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, and with origin $\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$. The matrix $\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is an element of $\mathrm{med}\,\big(\mathcal{G}_1(\mathbb{R}^2)\big)$.

On the set $\mathcal{M}(\mathbb{R}^m) \setminus \mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m))$, the projection on $\mathcal{G}_d(\mathbb{R}^m)$ is well-defined:

$$\begin{aligned}\mathrm{proj}\,(\cdot, \mathcal{G}_d(\mathbb{R}^m)) : \mathcal{M}(\mathbb{R}^m) \setminus \mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m)) &\longrightarrow \mathcal{G}_d(\mathbb{R}^m) \subset \mathcal{M}(\mathbb{R}^m) \\ A &\longmapsto P \text{ such that } \|P - A\|_{\mathrm{F}} = \mathrm{dist}\,(A, \mathcal{G}_d(\mathbb{R}^m))\,.\end{aligned}$$

The following lemma describes this projection explicitly. We defer its proof to Appendix A.

Lemma 2.1. *For any $A \in \mathcal{M}(\mathbb{R}^m)$, let A^s denote the matrix $A^s = \frac{1}{2}(A + {}^tA)$, and let $\lambda_1(A^s), ..., \lambda_n(A^s)$ be the eigenvalues of A^s in decreasing order. The distance from A to $\mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m))$ is*

$$\mathrm{dist}\,(A, \mathrm{med}\,(\mathcal{G}_d(\mathbb{R}^m))) = \frac{\sqrt{2}}{2}\big|\lambda_d(A^s) - \lambda_{d+1}(A^s)\big|.$$

If this distance is positive, the projection of A on $\mathcal{G}_d(\mathbb{R}^m)$ can be described as follows: consider the symmetric matrix A^s, and let $A^s = OD^tO$, with O an orthogonal matrix, and D the diagonal matrix containing the eigenvalues of A^s in decreasing order. Let J_d be the diagonal matrix whose first d terms are 1, and the other ones are zero. We have

$$\mathrm{proj}\left(A, \mathcal{G}_d(\mathbb{R}^m)\right) = OJ_d{}^tO.$$

Observe that, as a consequence of this lemma, every point of $\mathcal{G}_d(\mathbb{R}^m)$ is at equal distance from $\mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)$, and this distance is equal to $\frac{\sqrt{2}}{2}$. Therefore the reach of the subset $\mathcal{G}_d(\mathbb{R}^m) \subset \mathcal{M}(\mathbb{R}^m)$ is

$$\mathrm{reach}(\mathcal{G}_d(\mathbb{R}^m)) = \frac{\sqrt{2}}{2}.$$

Čech bundle filtration. Let X be a subset of $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Consider the usual Čech filtration $\mathbb{X} = (X^t)_{t\geq 0}$, where X^t denotes the t-thickening of $\check{X}$ in the metric space $(E, \|\cdot\|_\gamma)$. In order to give this filtration a vector bundle structure, consider the map p^t defined as the composition

$$X^t \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m) \xrightarrow{\quad\mathrm{proj}_2\quad} \mathcal{M}(\mathbb{R}^m) \setminus \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right) \xrightarrow{\mathrm{proj}(\cdot,\mathcal{G}_d(\mathbb{R}^m))} \mathcal{G}_d(\mathbb{R}^m), \tag{2}$$

where proj_2 represents the projection on the second coordinate of $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, and $\mathrm{proj}\left(\cdot, \mathcal{G}_d(\mathbb{R}^m)\right)$ the projection on $\mathcal{G}_d(\mathbb{R}^m) \subset \mathcal{M}(\mathbb{R}^m)$. Note that p^t is well-defined only when X^t does not intersect $\mathbb{R}^n \times \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)$. The supremum of such t's is denoted $t_\gamma^{\max}(X)$. We have

$$t_\gamma^{\max}(X) = \inf\left\{\mathrm{dist}_\gamma\left(x, \mathbb{R}^n \times \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)\right), x \in X\right\}, \tag{3}$$

where $\mathrm{dist}_\gamma\left(x, \mathbb{R}^n \times \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)\right)$ is the distance between the point $x \in \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ and the subspace $\mathbb{R}^n \times \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)$, with respect to the norm $\|\cdot\|_\gamma$. By definition of $\|\cdot\|_\gamma$, Equation 3 rewrites as

$$t_\gamma^{\max}(X) = \gamma \cdot \inf\{\mathrm{dist}\left(A, \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)\right), (y, A) \in X\},$$

where $\mathrm{dist}\left(A, \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)\right)$ represents the distance between the matrix A and the subspace $\mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)$ with respect to the Frobenius norm $\|\cdot\|_{\mathrm{F}}$. Denoting $t^{\max}(X)$ the value $t_\gamma^{\max}(X)$ for $\gamma = 1$, we obtain

$$\begin{aligned} & t_\gamma^{\max}(X) = \gamma \cdot t^{\max}(X) \\ \text{and} \qquad & t^{\max}(X) = \inf\{\mathrm{dist}\left(A, \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)\right), (y, A) \in X\}. \end{aligned} \tag{4}$$

Note that the values $t^{\max}(X)$ can be computed explicitly thanks to Lemma 2.1. In particular, if X is a subset of $\mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m)$, then $t^{\max}(X) = \frac{\sqrt{2}}{2}$. Accordingly,

$$t_\gamma^{\max}(X) = \frac{\sqrt{2}}{2}\gamma. \tag{5}$$

Definition 2.3 (Čech bundle filtration)**.** Consider a subset X of $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, and suppose that $t^{\max}(X) > 0$. The Čech bundle filtration associated to X in the ambient space $(E, \|\cdot\|_\gamma)$ is the vector bundle filtration $(\mathbb{X}, \mathbb{p})$ consisting of the Čech filtration $\mathbb{X} = (X^t)_{t\in T}$, and the maps $\mathbb{p} = (p^t)_{t\in T}$ as defined in Equation 2. This vector bundle filtration is defined on the index set $T = \left[0, t_\gamma^{\max}(X)\right)$, where $t_\gamma^{\max}(X)$ is defined in Equation 4.

The ith persistent Stiefel-Whitney class of the Čech bundle filtration $(\mathbb{X}, \mathbb{p})$, as in Definition 2.2, will be denoted $w_i(X)$ instead of $w_i(\mathbb{p})$.

Example 2.2. Let $E = \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$. Let X and Y be the subsets of E defined as:

$$X = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}$$
$$Y = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\frac{\theta}{2})^2 & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin(\frac{\theta}{2})^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}$$

The set X is to be seen as the normal bundle of the circle, and Y as the universal bundle of the circle, known as the Mobius band. We have $t^{\max}(X) = t^{\max}(Y) = \frac{\sqrt{2}}{2}$ as in Lemma 2.1. Let $\gamma = 1$.

Figure 10: Representation of the sets X and $Y \subset \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$: the black points correspond to the $\mathbb{R}^2$-coordinate, and the pink segments over them correspond to the orientation of the $\mathcal{M}(\mathbb{R}^2)$-coordinate.

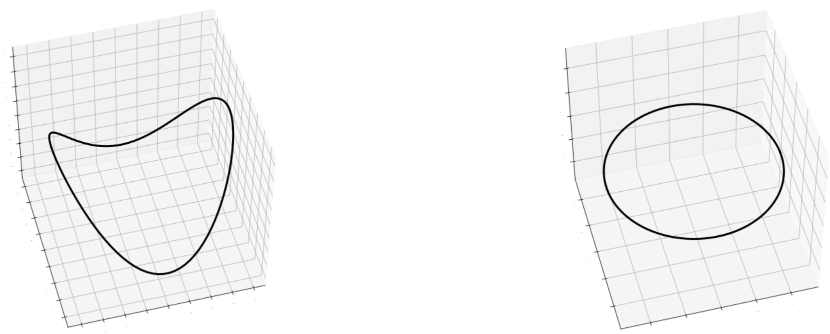


Figure 11: The sets X and $Y \subset \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$, projected in a 3-dimensional subspace of $\mathbb{R}^3$ via PCA.

We now compute the persistence diagrams of the Čech filtrations of X and Y in the ambient space E.

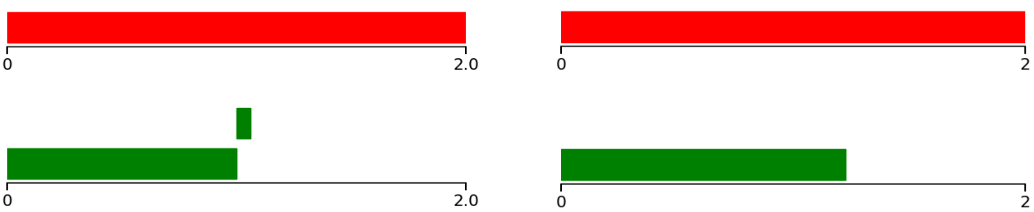


Figure 12: H^0 and H^1 persistence barcode of the Čech filtration of X (left) and Y (right).

Consider the first persistent Stiefel-Whitney classes $w_1(X)$ and $w_1(Y)$ of the corresponding Čech bundle filtrations. We compute that their lifebars are $\emptyset$ for $w_1(X)$, and $[0, t^{\max}(Y))$ for $w_1(Y)$. This is illustrated in Figure 13. One reads these bars as follows: $w_1^t(X)$ is zero for every $t \in \left[0, \frac{\sqrt{2}}{2}\right)$, while $w_1^t(Y)$ is nonzero.

Figure 13: Lifebars of the first persistent Stiefel-Whitney classes $w_1(X)$ and $w_1(Y)$.

2.3 Stability

In this subsection we derive a straigthforward stability result for persistent Stiefel-Whitney classes. We start by defining a notion of interleavings for vector bundle filtrations, in the same vein as the usual interleavings of set filtrations.

Definition 2.4 (Interleavings of vector bundle filtrations)**.** Let $\epsilon \geq 0$, and consider two vector bundle filtrations $(\mathbb{X}, \mathbb{p})$, $(\mathbb{Y}, \mathbb{q})$ of dimension d on E with respective index sets T and U. They are ϵ-interleaved if the underlying filtrations $\mathbb{X} = (X^t)_{t\in T}$ and $\mathbb{Y} = (Y^t)_{t\in U}$ are ϵ-interleaved, and if the following diagrams commute for every $t \in T \cap (U - \epsilon)$ and $s \in U \cap (T - \epsilon)$:

$$\begin{tikzcd} X^t \arrow[rr, hook] \arrow[dr, "p^t"'] & & Y^{t+\epsilon} \arrow[dl, "q^{t+\epsilon}"] \\ & \mathcal{G}_d(\mathbb{R}^\infty) & \end{tikzcd} \qquad \begin{tikzcd} Y^s \arrow[rr, hook] \arrow[dr, "q^s"'] & & X^{s+\epsilon} \arrow[dl, "q^{s+\epsilon}"] \\ & \mathcal{G}_d(\mathbb{R}^\infty) & \end{tikzcd}$$

The following theorem shows that interleavings of vector bundle filtrations give rise to interleavings of persistence modules which respect the persistent Stiefel-Whitney classes.

Theorem 2.3. *Consider two vector bundle filtrations $(\mathbb{X}, \mathbb{p})$, $(\mathbb{Y}, \mathbb{q})$ of dimension d with respective index sets T and U. Suppose that they are ϵ-interleaved. Then there exists an ϵ-interleaving (ϕ, ψ) between their corresponding persistent cohomology modules which sends persistent Stiefel-Whitney classes on persistent Stiefel-Whitney classes. In other words, for every $i \in [1, d]$, and for every $t \in (T + \epsilon) \cap U$ and $s \in U \cap (T + \epsilon)$, we have*

$$\begin{aligned} &\phi^t(w_i^t(\mathbb{p})) = w_i^{t-\epsilon}(\mathbb{q}) \\ \textit{and} \quad &\psi^s(w_i^s(\mathbb{p})) = w_i^{s-\epsilon}(\mathbb{q}). \end{aligned}$$

Proof. Define (ϕ, ψ) to be the ϵ-interleaving between the cohomology persistence modules $\mathbb{V}(\mathbb{X})$ and $\mathbb{V}(\mathbb{Y})$ given by the ϵ-interleaving between the filtrations $\mathbb{X}$ and $\mathbb{Y}$. Explicitly, if $i_t^{t+\epsilon}$ denotes the inclusion $X^t \hookrightarrow Y^{t+\epsilon}$ and $j_s^{s+\epsilon}$ denotes the inclusion $Y^s \hookrightarrow X^{s+\epsilon}$, then $\phi = (\phi^t)_{t\in(T+\epsilon)\cap U}$ is given by the induced maps in cohomology $\phi^t = (i_{t-\epsilon}^t)^*$, and $\psi = (\psi^s)_{s\in(U+\epsilon)\cap T}$ is given by $\psi^s = (j_{s-\epsilon}^s)^*$.

Now, by fonctoriality, the diagrams of Definition 2.4 give rise to commutative diagrams in cohomology:

$$\begin{tikzcd} H^*(X^{t-\epsilon}) & & H^*(Y^t) \arrow[ll, "\phi^t"'] \\ & H^*(\mathcal{G}_d(\mathbb{R}^\infty)) \arrow[ul, "(p^{t-\epsilon})^*"] \arrow[ur, "(q^t)^*"'] & \end{tikzcd} \qquad \begin{tikzcd} H^*(Y^{s-\epsilon}) & & H^*(X^s) \arrow[ll, "\psi^s"'] \\ & H^*(\mathcal{G}_d(\mathbb{R}^\infty)) \arrow[ul, "(q^{s-\epsilon})^*"] \arrow[ur, "(p^s)^*"'] & \end{tikzcd}$$

Let $i \in [1, d]$. By definition, the persistent Stiefel-Whitney classes $w_i(\mathbb{p}) = (w_i^t(\mathbb{p}))_{t\in T}$ and $w_i(\mathbb{q}) = (w_i^s(\mathbb{q}))_{s\in U}$ are $w_i^t(\mathbb{p}) = (p^t)^*(w_i)$ and $w_i^s(\mathbb{q}) = (q^s)^*(w_i)$, where w_i is the ith Stiefel-Whitney class of $\mathcal{G}_d(\mathbb{R}^\infty)$. The previous commutative diagrams then tranlates as $\phi^t(w_i^t(\mathbb{p})) = w_i^{t-\epsilon}(\mathbb{q})$ and $\psi^s(w_i^s(\mathbb{p})) = w_i^{s-\epsilon}(\mathbb{q})$, as wanted. $\square$

Consider two vector bundle filtrations $(\mathbb{X}, \mathbb{p})$, $(\mathbb{Y}, \mathbb{q})$ such that there exists an ϵ-interleaving (ϕ, ψ) between their persistent cohomology modules $\mathbb{V}(\mathbb{X})$, $\mathbb{V}(\mathbb{Y})$ which sends persistent Stiefel-Whitney classes on persistent Stiefel-Whitney classes. Let $i \in [1, d]$. Then the lifebars of their ith persistent Stiefel-Whitney classes $w_i(\mathbb{p})$ and $w_i(\mathbb{q})$ are ϵ-close in the following sense: if we denote $t^\dagger(\mathbb{p}) = \inf\{t \in T, w_i^t(\mathbb{p}) \neq 0\}$ and $t^\dagger(\mathbb{q}) = \inf\{t \in T, w_i^t(\mathbb{q}) \neq 0\}$, then $|t^\dagger(\mathbb{p}) - t^\dagger(\mathbb{q})| \leq \epsilon$.

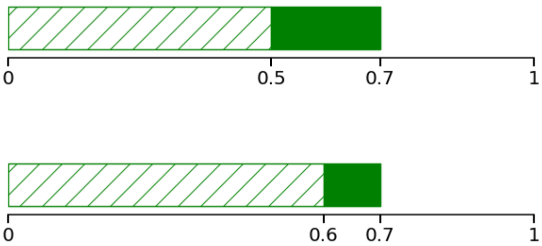


Figure 14: Two ϵ-close lifebars, with $\epsilon = 0{,}1$.

Let us apply this result to the Čech bundle filtrations. Let X and Y be two subsets of $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Suppose that the Hausdorff distance $d_{\mathrm{H}}(X, Y)$, with respect to the norm $\|\cdot\|_\gamma$, is not greater than ϵ, meaning that the ϵ-thickenings X^ϵ and Y^ϵ satisfiy $Y \subseteq X^\epsilon$ and $X \subseteq Y^\epsilon$. It is then clear that the vector bundle filtrations are ϵ-interleaved, and we can apply Theorem 2.3 to obtain the following result.

Corollary 2.4. *If two subsets $X, Y \subset E$ satisfy $d_{\mathrm{H}}(X, Y) \leq \epsilon$, then there exists an ϵ-interleaving between the persistent cohomology modules of their corresponding Čech bundle filtrations which sends persistent Stiefel-Whitney classes on persistent Stiefel-Whitney classes.*

Example 2.5. In order to illustrate Corollary 2.4, consider the sets X' and Y' represented in Figure 15. They are noisy samples of the sets X and Y defined in Example 2.2. They contain 50 points each.

Figure 15: Representation of the sets $X', Y' \subset \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$.

We choose the parameter $\gamma = 1$. Figure 16 represents the barcodes of the Čech filtrations of the sets X' and Y', together with the lifebar of the first persistent Stiefel-Whitney class of their corresponding Čech bundle filtrations. Observe that they are close to the original descriptors of X and Y (Figures 12 and 13).

Experimentally, we computed that the Hausdorff distances between X, X' and Y, Y' are approximately $d_{\mathrm{H}}(X, X') \approx 0{,}43$ and $d_{\mathrm{H}}(Y, Y') \approx 0{,}39$. This is coherent with the lifebar of $w_1(Y')$, which is ϵ-close to the lifebar of $w_1(Y)$ with $\epsilon \approx 0{,}3 \leq 0{,}39$.

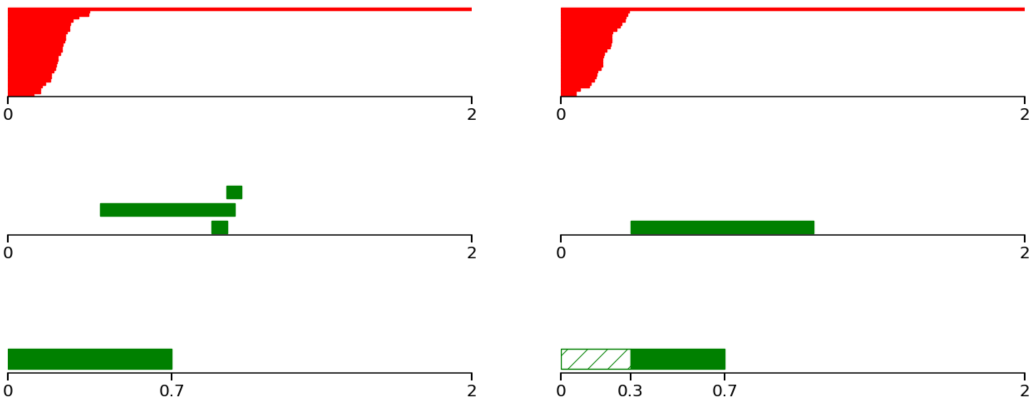


Figure 16: Left: H^0 and H^1 barcodes of X' and lifebar of $w_1(X')$. Right: same for Y'.

2.4 Consistency

In this subsection we describe a setting where the persistent Stiefel-Whitney classes $w_i(X)$ of the Čech bundle filtration of a set X can be seen as consistent estimators of the Stiefel-Whitney classes of some underlying vector bundle.

Let $\mathcal{M}_0$ be a compact $\mathcal{C}^2$-manifold, and $u_0 \colon \mathcal{M}_0 \to \mathbb{R}^n$ an immersion. Suppose that $\mathcal{M}_0$ is given a d-dimensional vector bundle structure $p \colon \mathcal{M}_0 \to \mathcal{G}_d(\mathbb{R}^m)$. Let $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, and consider the set

$$\mathcal{M} = \left\{ \left(u_0(x_0), P_{p(x_0)}\right), x_0 \in \mathcal{M}_0 \right\} \subset E, \tag{6}$$

where $P_{p(x_0)}$ denotes the orthogonal projection matrix onto the subspace $p(x_0) \subset \mathbb{R}^m$. The set $\mathcal{M}$ is called the *lift* of $\mathcal{M}_0$. Consider the *lifting map* defined as

$$\begin{aligned} u \colon \mathcal{M}_0 &\longrightarrow \mathcal{M} \subset E \\ x_0 &\longmapsto \left(u_0(x_0), P_{p(x_0)}\right). \end{aligned} \tag{7}$$

We make the following assumption: u is an embedding. As a consequence, $\mathcal{M}$ is a submanifold of E, and $\mathcal{M}_0$ and $\mathcal{M}$ are diffeomorphic. An extensive study of this setting can be found in [Tin19].

The persistent cohomology of $\mathcal{M}$ can be used to recover the cohomology of $\mathcal{M}_0$. To see this, select $\gamma > 0$, and denote by $\mathrm{reach}(\mathcal{M})$ the reach of $\mathcal{M}$, where E is endowed with the norm $\|\cdot\|_\gamma$. Note that $\mathrm{reach}(\mathcal{M})$ is positive and depends on γ. Let $\mathbb{M} = (\mathcal{M}^t)_{t\geq 0}$ be the Čech set filtration of $\mathcal{M}$ in the ambient space $(E, \|\cdot\|_\gamma)$, and let $\mathbb{V}(\mathcal{M})$ be the corresponding persistent cohomology module. For every $s, t \in [0, \mathrm{reach}(\mathcal{M}))$ such that $s \leq t$, we know that the inclusion maps $i_s^t \colon \mathcal{M}^s \hookrightarrow \mathcal{M}^t$ are homotopy equivalences (see Subsection 1.3). Hence the persistence module $\mathbb{V}(\mathcal{M})$ is constant on the interval $[0, \mathrm{reach}(\mathcal{M}))$, and is equal to the cohomology $H^*(\mathcal{M}) = H^*(\mathcal{M}_0)$.

Consider the Čech bundle filtration $(\mathbb{M}, \mathbb{p})$ of $\mathcal{M}$. The following theorem shows that the persistent Stiefel-Whitney classes $w_i^t(\mathcal{M})$ are also equal to the usual Stiefel-Whitney classes of the vector bundle $(\mathcal{M}_0, p)$.

Theorem 2.6. *Let $\mathcal{M}_0$ be a compact $\mathcal{C}^2$-manifold, $u_0 \colon \mathcal{M}_0 \to \mathbb{R}^n$ an immersion and $p \colon \mathcal{M}_0 \to \mathcal{G}_d(\mathbb{R}^m)$ a continuous map. Let $\mathcal{M}$ be the lift of $\mathcal{M}_0$ (Equation 6) and u the lifting map (Equation 7). Suppose that u is an embedding.*

Let $\gamma > 0$ and consider the Čech bundle filtration $(\mathbb{M}, \mathbb{p})$ of $\mathcal{M}$. Its maximal filtration value is $t_\gamma^{\max}(\mathcal{M}) = \frac{\sqrt{2}}{2}\gamma$. Denote by $w_i(\mathbb{p}) = (w_i^t(\mathbb{p}))_{t \in T}$ its persistent Stiefel-Whitney classes, $i \in [1, d]$. Denote also by i_0^t the inclusion $\mathcal{M} \to \mathcal{M}^t$, for $t \in [0, \mathrm{reach}(\mathcal{M}))$.

Let $t \geq 0$ be such that $t < \min\left(\mathrm{reach}(\mathcal{M}), t_\gamma^{\max}(\mathcal{M})\right)$. Then the map $i_0^t \circ u \colon \mathcal{M}_0 \to \mathcal{M}^t$ induces an isomorphism $H^*(\mathcal{M}_0) \leftarrow H^*(\mathcal{M}^t)$ which maps the ith persistent Stiefel-Whitney class $w_i^t(\mathbb{p})$ of $(\mathbb{M}, \mathbb{p})$ to the ith Stiefel-Whitney class of $(\mathcal{M}_0, p)$.

Proof. Consider the following commutative diagram, defined for every $t < t_\gamma^{\max}(\mathcal{M})$:

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "u"] \arrow[dr, "p"'] & \mathcal{M} \arrow[r, hook, "i_0^t"] & \mathcal{M}^t \arrow[dl, "p^t"] \\ & \mathcal{G}_d(\mathbb{R}^m) & \end{tikzcd}$$

We obtain a commutative diagram in cohomology:

$$\begin{tikzcd} H^*(\mathcal{M}_0) & H^*(\mathcal{M}) \arrow[l, "u^*"'] & H^*(\mathcal{M}^t) \arrow[l, "(i_0^t)^*"'] \\ & H^*(\mathcal{G}_d(\mathbb{R}^m)) \arrow[ul, "p^*"] \arrow[ur, "(p^t)^*"'] & \end{tikzcd}$$

Since $t < \mathrm{reach}(\mathcal{M})$, the map $(i_0^t)^*$ is an isomorphism (see Subsection 1.3). So is u^* since u is an embedding. As a consequence, the map $i_0^t \circ u$ induces an isomorphism $H^*(\mathcal{M}_0) \simeq H^*(\mathcal{M}^t)$.

Let w_i denotes the ith Stiefel-Whitney class of $\mathcal{G}_d(\mathbb{R}^m)$. By definition, the ith Stiefel-Whitney class of $(\mathcal{M}_0, p)$ is $p^*(w_i)$, and the ith persistent Stiefel-Whitney class of $(\mathbb{M}, \mathbb{p})$ is $w_i^t(\mathbb{p}) = (p^t)^*(w_i)$. By commutativity of the diagram, we obtain $p^*(w_i) = (p^t)^*(w_i)$, under the identification $H^*(\mathcal{M}_0) \simeq H^*(\mathcal{M}^t)$.

□

Applying Theorems 2.3, 2.6 and the considerations of Subsection 1.3 yield an estimation result.

Corollary 2.7. *Let $X \subset E$ be any subset such that $d_{\mathrm{H}}(X, \mathcal{M}) \leq \epsilon$. Then for every $t \in [4\epsilon, \mathrm{reach}(\mathcal{M}) - 3\epsilon)$, the composition of inclusions $\mathcal{M}_0 \hookrightarrow \mathcal{M} \hookrightarrow X^t$ induces an isomorphism $H^*(\mathcal{M}_0) \leftarrow H^*(X^t)$ which sends the ith persistent Stiefel-Whitney class $w_i^t(X)$ of the Čech bundle filtration of X to the ith Stiefel-Whitney class of $(\mathcal{M}_0, p)$.*

As a consequence of this corollary, on the set $[4\epsilon, \mathrm{reach}(\mathcal{M}) - 3\epsilon)$, the ith persistent Stiefel-Whitney class of the Čech bundle filtration of X is zero if and only if the ith Stiefel-Whitney class of $(\mathcal{M}_0, p)$ is.

Example 2.8. In order to illustrate Corollary 2.7, consider the torus and the Klein bottle, immersed in $\mathbb{R}^3$ as in Figure 17.

Figure 17: Immersion of the torus and the Klein bottle in $\mathbb{R}^3$.

Let them be endowed with their normal bundles. They can be seen as submanifolds $\mathcal{M}, \mathcal{M}'$ of $\mathbb{R}^3 \times \mathcal{M}(\mathbb{R}^3)$. We consider two samples X, X' of $\mathcal{M}, \mathcal{M}'$, represented in Figure 18. They contain respectively 346 and 1489 points. We computed experimentally the Hausdorff distances $d_{\mathrm{H}}(X, \mathcal{M}) \approx 0{,}6$ and $d_{\mathrm{H}}(X', \mathcal{M}') \approx 0{,}45$, with respect to the norm $\|\cdot\|_\gamma$ where $\gamma = 1$.

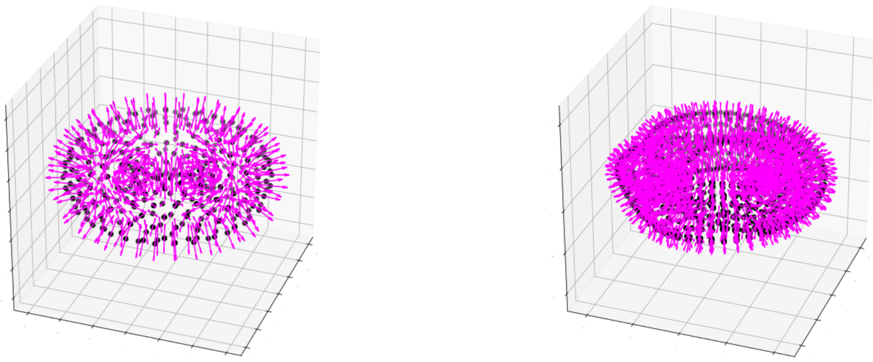


Figure 18: Samples X and X' of $\mathcal{M}$ and $\mathcal{M}'$. The black points corresponds to the $\mathbb{R}^3$-coordinate, and the pink arrows over them correspond to the orientation of the $\mathcal{M}(\mathbb{R}^3)$-coordinate.

Figure 19 represents the barcodes of the persistent cohomology of X and X', and the lifebars of their first persistent Stiefel-Whitney classes $w_1(X)$ and $w_1(X')$. Observe that $w_1(X)$ is always zero, while $w_1(X')$ is nonzero for $t \geq 0{,}3$. This is an indication that $\mathcal{M}$, the underlying manifold of X, is orientable, while $\mathcal{M}'$ is not. To see this, recall Proposition 1.4: the first Stiefel-Whitney class of the tangent bundle of a manifold is zero if and only if the manifold is orientable. One can deduce the following fact: the first Stiefel-Whitney class of the normal bundle of an immersed manifold is zero if and only if the manifold is orientable (see the following lemma). Therefore, one interprets these lifebars as follows: X is sampled on an orientable manifold, while X' is sampled on a non-orientable one.

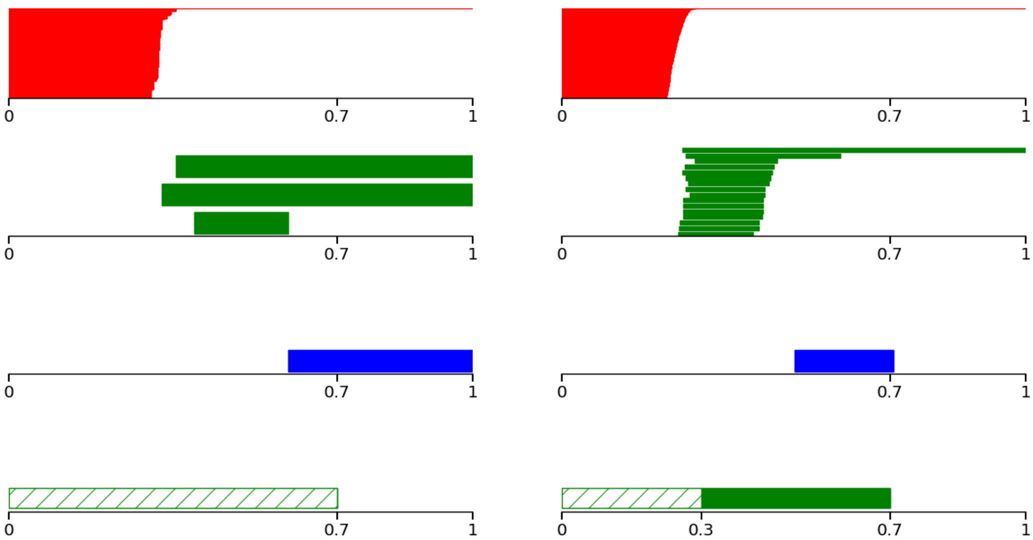


Figure 19: Left: H^0, H^1 and H^2 barcodes of X and lifebar of $w_1(X)$. Right: same for X'.

Lemma 2.9. *Let $\mathcal{M}_0 \to \mathcal{M}$ be an immersion of a manifold $\mathcal{M}_0$ in a Euclidean space. Then $\mathcal{M}_0$ is orientable if and only if the first Stiefel-Whitney class of its normal bundle is zero.*

Proof. Let τ and ν denote the tangent and normal bundles of $\mathcal{M}_0$. The Whitney sum $\tau \oplus \nu$ is a trivial bundle, hence its first Stiefel-Whitney class is $w_1(\tau \oplus \nu) = 0$. Using Axioms 1 and 3 of the Stiefel-Whitney classes, we obtain

$$\begin{aligned} w_1(\tau \oplus \nu) &= w_1(\tau) \smile w_0(\nu) + w_0(\tau) \smile w_1(\nu) \\ &= w_1(\tau) \smile 1 + 1 \smile w_1(\nu) \\ &= w_1(\tau) + w_1(\nu). \end{aligned}$$

Therefore, $w_1(\tau) = w_1(\nu)$. To conclude, $w_1(\tau)$ is zero if and only if $w_1(\nu)$ is zero, and Proposition 1.4 yields the result. □

3 Simplicial approximation of Čech bundle filtrations

In order to build an effective algorithm to compute the persistent Stiefel-Whitney classes, we have to find an equivalent formulation in terms of simplicial cohomology. We start by reviewing the usual technique of simplicial approximation, and then apply it to the particular case of Čech bundle filtrations.

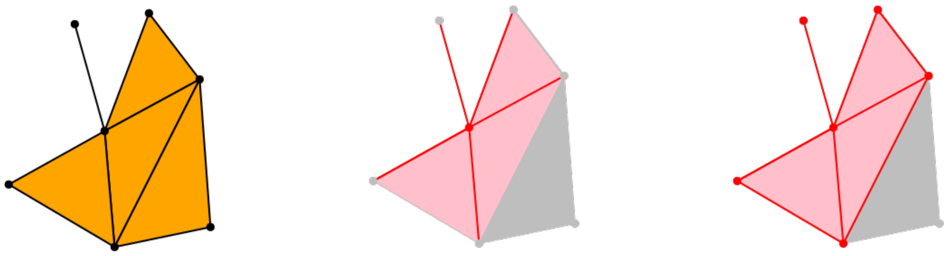
3.1 Background on simplicial complexes

To start, we recall some elements of combinatorics and topology of simplicial complexes.

(Combinatorial) simplicial complexes. Let K be a simplicial complex. It means that there exists a set V, the set of *vertices*, such that $K \subseteq \mathcal{P}(V)$, and K satisfies the following condition: for every $\sigma \in K$ and every subset $\nu \subseteq \sigma$, ν is in K. The elements of K are called *faces* or *simplices* of the simplicial complex K.

For every simplex $\sigma \in K$, we define its dimension $\dim(\sigma) = \mathrm{card}(\sigma) - 1$. The dimension of K, denoted $\dim(K)$, is the maximal dimension of its simplices. For every $i \geq 0$, the i-skeleton K^i is defined as the subset of K consisting of simplices of dimension at most i. Note that K^0 corresponds to the underlying vertex set V, and K^1 is a graph.

Given a simplex $\sigma \in K$, its (open) star $\mathrm{St}(\sigma)$ is the set of all the simplices $\nu \in K$ that contain σ. The open star is not a simplicial complex in general. We also define its closed star $\overline{\mathrm{St}}(\sigma)$ as the smallest simplicial subcomplex of K which contains $\mathrm{St}(\sigma)$.



K $\qquad$ $\mathrm{St}(v)$ in red and pink $\qquad$ $\overline{\mathrm{St}}(v)$ in red and pink

Figure 20: Open and closed star of a vertex of K.

Given a graph G, the corresponding clique complex is the simplicial complex whose simplices are the sets of vertices of the cliques of G. We say that a simplicial complex K is a flag complex if it is the clique complex of its 1-skeleton K^1.

Topological realizations. For every $p \geq 0$, the standard p-simplex Δ^p is a topological space defined as the convex hull of the canonical basis vectors $e_1, ..., e_{p+1}$ of $\mathbb{R}^{p+1}$, endowed with the subspace topology. We now describe the construction of the *topological realization* of the simplicial complex K, denoted $|K|$. It is a particular case of the construction CW-complexes [Hat02, Appendix].

1. Start with the discrete topological space $\left|K^0\right|$ consisting of the vertices of K.

2. Inductively, form the p-skeleton $|K^p|$ from $\left|K^{p-1}\right|$ by attaching p-dimensional simplices to $\left|K^{p-1}\right|$. More precisely, for each $\sigma \in K$ of dimension p, take a copy of the standard p-simplex Δ^p. Denote this simplex by $\Delta\sigma$. Label its vertices with the elements of σ. Whenever $\tau \subset \sigma \in K$, identify $\Delta\tau$ with a subset of $\Delta\sigma$, via the face inclusion which sends the elements of τ to the corresponding elements of σ. Give $|K^p|$ the quotient topology.

3. Endow $|K| = \bigcup_{p\geq 0} |K^p|$ with the weak topology: a set $A \subset |K|$ is open if and only if $A \cap |K^p|$ is open in $|K^p|$ for each $p \geq 0$.

Alternatively, the topology on $|K|$ can be described as follows: a subset $A \subset |K|$ is open (or closed) if and only if for every $\sigma \in K$, the set $A \cap \Delta\sigma$ is open (or closed) in $\Delta\sigma$. Note that condition 3 is superfluous when K is finite dimensional.

If $\sigma = [v]$ is a vertex of K, we will denote by $|\sigma|$ the singleton $\{v\}$, seen as a subset of $|K|$. If σ is a face of K of dimension at least 1, we will denote by $|\sigma|$ the open subset of $|K|$ which corresponds to the interior of the face $\Delta\sigma \subset |K|$. We denote by $\overline{|\sigma|}$ its closure in $|K|$. Observe that if $\overline{\sigma}$ denotes the smallest simplicial subcomplex of K that contains σ, then $|\overline{\sigma}| = \Delta\sigma = \overline{|\sigma|}$. The following set is a partition of $|K|$:

$$\{|\sigma|\,, \sigma \in K\}\,.$$

This allows to define the *face map* of K. It is the unique map $\mathcal{F}_K \colon |K| \to K$ that satisfies $x \in |\mathcal{F}_K(x)|$ for every $x \in |K|$.

If L is a subset of K, we define its topological realization as $|L| = \bigcup_{\sigma\in L} |\sigma|$. For every simplex $\sigma \in K$, the topological realization of its open star, $|\mathrm{St}\,(\sigma)|$, is open in $|K|$. Besides, the topological realization of its closed star, $\left|\overline{\mathrm{St}}\,(\sigma)\right|$, is equal to $\overline{|\mathrm{St}\,(\sigma)|}$, hence is closed with respect to the weak topology.

If σ is a face of K of dimension at least 1, the subset $|\sigma|$ of $|K|$ is canonically homeomorphic to the interior of the standard p-simplex Δ^p, where $p = \dim(\sigma)$. This allows to define on $|K|$ the barycentric coordinates: for every face $\sigma = [v_0, ..., v_p] \in K$, the points $x \in |\sigma|$ can be written as

$$x = \sum_{i=0}^{p} \lambda_i v_i$$

with $\lambda_0, ..., \lambda_p > 0$ and $\sum_{i=0}^{p} \lambda_i = 1$.

Triangulation and geometric realizations. Let X be a subset of E. A triangulation of X consists of a simplicial complex K together with a homeomorphism $h \colon X \to |K|$. The set X is called a *geometric realization* of K, and K is called a *triangulation* of X.

3.2 Simplicial approximation

This subsection is based on [Hat02, Section 2.C]. In the following, K and L are two simplicial complexes. We recall the reader that $|K|$ denotes the topological realization of K, and $\mathrm{St}\,(v)\,, \overline{\mathrm{St}}\,(v)$ denote the open and closed star of a vertex $v \in K^0$.

Simplicial maps. A *simplicial map* between simplicial complexes K and L is a map $g \colon |K| \to |L|$ which sends vertices on vertices and is linear on every simplices. In other words, for every $\sigma = [v_0, ..., v_p] \in K$, the map g restricted to $|\sigma| \subset |K|$ can be written in barycentric coordinates as

$$\sum_{i=0}^{p} \lambda_i v_i \;\longmapsto\; \sum_{i=0}^{p} \lambda_i g(v_i). \tag{8}$$

A simplicial map $g\colon |K| \to |L|$ is uniquely determined by its restriction to the vertex sets $g_{|K^0}\colon K^0 \to L^0$. Reciprocally, let $f\colon K^0 \to L^0$ be a map between vertex sets which satisfies the following condition:

$$\forall \sigma \in K, f(\sigma) \in L. \tag{9}$$

Then f induces a simplicial map via barycentric coordinates, denoted $|f|\colon |K| \to |L|$. In the rest of the paper, a simplicial map will either refer to a map $g\colon |K| \to |L|$ which satisfies Equation 8, to a map $f\colon K^0 \to L^0$ which satisfies Equation 9, or to the induced map $f\colon K \to L$.

Simplicial approximation. Let $g\colon |K| \to |L|$ be any continuous map. The problem of the simplicial approximation consists in finding a simplicial map $f\colon K \to L$ with topological realization $|f| : |K| \to |L|$ homotopic to g. A way to solve this problem is to consider the following property: we say that the map g satisfies the *star condition* if for every vertex v of K, there exists a vertex w of L such that

$$g\left(\left|\overline{\mathrm{St}}\,(v)\right|\right) \subseteq \left|\mathrm{St}\,(w)\right|.$$

If this is the case, let $f\colon K^0 \to L^0$ be any map between vertex sets such that for every vertex v of K, we have $g\left(\left|\overline{\mathrm{St}}\,(v)\right|\right) \subseteq |\mathrm{St}\,(f(v))|$. Equivalently, f satisfies

$$g\left(\overline{\mathrm{St}}\,(v)\right) \subseteq \mathrm{St}\,(f(v)).$$

Such a map is called a *simplicial approximation to g*. One shows that it is a simplicial map, and that its topological realization $|f|$ is homotopic to g [Hat02, Theorem 2C.1].

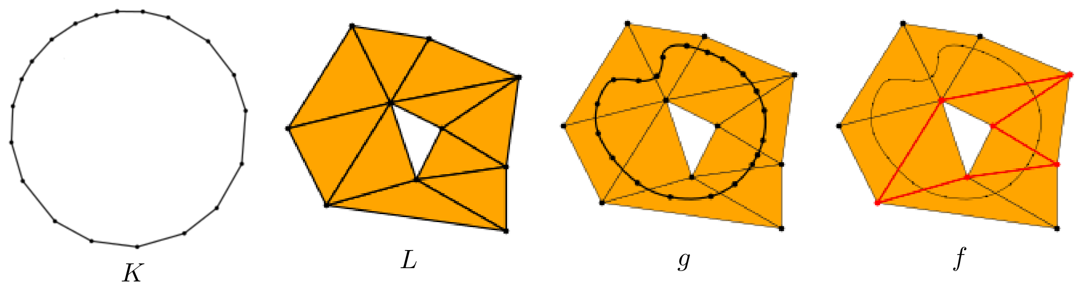


Figure 21: The map f (in red) is a simplicial approximation to g.

In general, a map g may not satisfy the star condition. However, there is always a way to subdivise the simplicial complex K in order to obtain an induced map which does (see Theorem 3.1). We describe this construction in the following paragraph.

Barycentric subdivisions. Let us describe briefly the process of barycentric subdivision of a simplicial complex. A more extensive description can be found in [Hat02, Proof of Proposition 2.21]. Let Δ^p denote the standard p-simplex, with vertices denoted $v_0, ..., v_p$. The barycentric subdivision of Δ^p consists in decomposing Δ^p into $(p+1)!$ simplices of dimension p. It is a simplicial complex, whose vertex set corresponds to the points $\sum_{i=0}^{p} \lambda_i v_i$ for which some λ_i are zero and the other ones are equal. Equivalently, one can see these this new set of vertices as a the power set of the set of vertices of Δ^p.

More generally, if K is a simplicial complex, its barycentric subdivision $\mathrm{sub}(K)$ is the simplicial complex obtained by subdivising each of its faces. The set of vertices of $\mathrm{sub}(K)$ can be seen as a subset of the power set of the set of vertices of K.

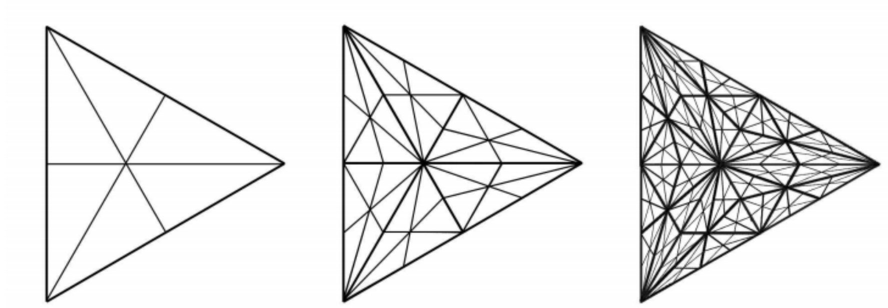


Figure 22: The first three barycentric subdivisions of a 2-simplex.

By subdivising K, we shrink its faces. More precisely, if $h\colon X \to |K|$ is a geometric realization of K, with $X \subset \mathbb{R}^n$, and if D is the diameter of a face $\sigma \in K$ seen in X, then the faces of the barycentric subdivision of σ are of diameter at most $\frac{\dim(\sigma)}{\dim(\sigma)+1} D$. Therefore one can repeat the subdivision to obtain arbitrarly small faces. Applying barycentric subdivisions n times will be denoted $\mathrm{sub}^n(K)$.

Theorem 3.1 ([Hat02, Theorem 2C.1])**.** *Consider two simplicial complexes K, L with K finite, and let $g\colon |K| \to |L|$ be a continuous map. Then there exists $n \geq 0$ such that $g\colon |\mathrm{sub}^n(K)| \to |L|$ satisfies the star condition.*

As a consequence, such a map $g\colon |\mathrm{sub}^n(K)| \to |L|$ admits a simplicial approximation. This is known as the simplical approximation theorem. As an illustration, Figure 23 represents a map $g\colon |K| \to |L|$ which does not satisfies the star condition, but whose first barycentric subdivision does.

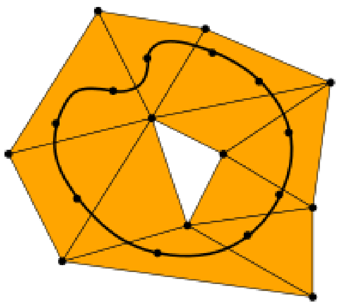


Figure 23: The map $g\colon |K| \to |L|$ does not satisfy the star condition, but its first barycentric subdivision does (see Figure 21).

3.3 Application to Čech bundle filtrations

In this subsection, we apply the principle of simplicial approximation to the particular case of persistent Stiefel-Whitney classes of Čech bundle filtrations.

Let X be a subset of $E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Let us recall Definition 2.3: the Čech bundle filtration associated to X is the vector bundle filtration $(\mathbb{X}, \mathbb{p})$ whose underlying filtration is the Čech filtration $\mathbb{X} = (X^t)_{t\in T}$, with $T = [0, t_\gamma^{\max}(X))$, and whose maps $\mathbb{p} = (p^t)_{t\in T}$ are given by the following composition, as in Equation 2:

$$\begin{tikzcd} X^t \arrow[r, "\mathrm{proj}_2"] \arrow[rr, bend right, "p^t"'] & \mathcal{M}(\mathbb{R}^m) \setminus \mathrm{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right) \arrow[r, "\mathrm{proj}(\cdot, \mathcal{G}_d(\mathbb{R}^m))"] & \mathcal{G}_d(\mathbb{R}^m). \end{tikzcd}$$

Let $t \in T$. The aim of this subsection is to describe a simplicial approximation to $p^t : X^t \to \mathcal{G}_d(\mathbb{R}^m)$. To do so, let us fix a triangulation L of $\mathcal{G}_d(\mathbb{R}^m)$. It comes with a homeomorphism $h : \mathcal{G}_d(\mathbb{R}^m) \to |L|$. We will now triangulate the Čech set filtration X^t, as described in Subsection 1.3. The thickening X^t is a subset of the metric space $(E, \|\cdot\|_\gamma)$ which consists in a union of closed balls centered around points of X:

$$X^t = \bigcup_{x \in X} \overline{\mathcal{B}}_\gamma (x, t),$$

where $\overline{\mathcal{B}}_\gamma (x, t)$ denotes the closed ball of center x and radius t for the norm $\|\cdot\|_\gamma$. Let $\mathcal{U}^t$ denote the cover $\{\overline{\mathcal{B}}_\gamma (x, t), x \in X\}$ of X^t, and let $\mathcal{N}(\mathcal{U}^t)$ be its nerve. By the nerve theorem for convex closed covers [BCY18, Theorem 2.9], the simplicial complex $\mathcal{N}(\mathcal{U}^t)$ is homotopy equivalent to its underlying set X^t. That is to say, there exists a continuous map $g^t : |\mathcal{N}(\mathcal{U}^t)| \to X^t$ which is a homotopy equivalence.

As a consequence, in cohomological terms, the map $p^t : X^t \to \mathcal{G}_d(E)$ is equivalent to the map q^t defined as $q^t = h \circ p^t \circ g^t$.

$$\begin{tikzcd}
X^t \arrow[r, "p^t"] & \mathcal{G}_d(\mathbb{R}^m) \arrow[d, "h"] \\
|\mathcal{N}(\mathcal{U}^t)| \arrow[u, "g^t"] \arrow[r, dashed, "q^t"] & |L|
\end{tikzcd} \tag{10}$$

This gives a way to compute the induced map $(p^t)^* : H^*(X^t) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^m))$ algorithmically:

- Subdivise $\mathcal{N}(\mathcal{U}^t)$ until q^t satisfies the star condition (as in Theorem 3.1),
- Choose a simplicial approximation f^t to q^t,
- Compute the induced map between simplicial cohomology groups $(f^t)^* : H^*(\mathcal{N}(\mathcal{U}^t)) \leftarrow H^*(L)$.

By correspondance between simplicial and singular cohomology, the map $(f^t)^*$ corresponds to $(p^t)^*$. Hence the problem of computing $(p^t)^*$ is solved, if it were not for the following issue: in practice, the map $g^t : |\mathcal{N}(\mathcal{U}^t)| \to X^t$ given by the nerve theorem is not explicit. The rest of this subsection is devoted to showing that g^t can be chosen canonically as the *shadow map*.

Shadow map. We still consider the thickening X^t, the corresponding cover $\mathcal{U}^t$ and its nerve $\mathcal{N}(\mathcal{U}^t)$. The underlying vertex set of the simplicial complex $\mathcal{N}(\mathcal{U}^t)$ is the set X itself. The shadow map $g^t : |\mathcal{N}(\mathcal{U}^t)| \to X^t$ is defined as follows: for every simplex $\sigma = [x_0, ..., x_p] \in \mathcal{N}(\mathcal{U}^t)$ and every point $\sum_{i=0}^p \lambda_i x_i$ of $|\sigma|$ written in barycentric coordinates, associate the point $\sum_{i=0}^p \lambda_i x_i$ of E:

$$g^t : \sum_{i=0}^p \lambda_i x_i \in |\sigma| \longmapsto \sum_{i=0}^p \lambda_i x_i \in E.$$

The author is not aware if the shadow map is indeed a homotopy equivalence from $|\mathcal{N}(\mathcal{U}^t)|$ to X^t. Nevertheless, the following result will be enough for our purposes: the shadow map induces an isomorphism at cohomology level.

Lemma 3.2. *Suppose that X is finite and in general position. Then the shadow map $g^t : |\mathcal{N}(\mathcal{U}^t)| \to X^t$ induces an isomorphism $(g^t)^* : H^*(|\mathcal{N}(\mathcal{U}^t)|) \leftarrow H^*(X^t)$.*

Proof. Recall that $\mathcal{U}^t = \left\{\overline{\mathcal{B}}_\gamma (x, t), x \in X\right\}$. Let us consider a smaller cover. For every $x \in X$, let $\mathrm{Vor}(x)$ denote the Voronoi cell of x in the ambient metric space $(E, \|\cdot\|_\gamma)$, and define

$$\mathcal{V}^t = \left\{\overline{\mathcal{B}}(x, t) \cap \mathrm{Vor}(x), x \in X\right\}.$$

The set $\mathcal{V}^t$ is a cover of X^t, and its nerve $\mathcal{N}(\mathcal{V}^t)$ is known as the Delaunay complex (see [BE17]). Let $h^t : |\mathcal{N}(\mathcal{V}^t)| \to X^t$ denote the shadow map of $\mathcal{N}(\mathcal{V}^t)$. The Delaunay complex is a subcomplex of the Čech complex, hence we can consider the following diagram:

$$\begin{tikzcd}
{|\mathcal{N}(\mathcal{V}^t)|} \arrow[r, hook] \arrow[rr, bend left, "h^t"] & {|\mathcal{N}(\mathcal{U}^t)|} \arrow[r, "g^t"'] & X^t .
\end{tikzcd}$$

This yields the following commutative diagram between cohomology rings:

$$\begin{tikzcd}
H^*(|\mathcal{N}(\mathcal{V}^t)|) & H^*(|\mathcal{N}(\mathcal{U}^t)|) \arrow[l] & H^*(X^t) . \arrow[l, "(g^t)^*"] \arrow[ll, bend right, "(h^t)^*"']
\end{tikzcd}$$

Now, it is proven in [Ede93, Theorem 3.2] that the shadow map $h^t\colon |\mathcal{N}(\mathcal{V}^t)| \to X^t$ is a homotopy equivalence (it is required here that X is in general position). Therefore the map $(h^t)^*\colon H^*(|\mathcal{N}(\mathcal{V}^t)|) \leftarrow H^*(X^t)$ is an isomorphism. Moreover, we know from [BE17, Theorem 5.10] that $\mathcal{N}(\mathcal{U}^t)$ collapses to $\mathcal{N}(\mathcal{V}^t)$. Therefore the inclusion $|\mathcal{N}(\mathcal{V}^t)| \hookrightarrow |\mathcal{N}(\mathcal{U}^t)|$ also is a homotopy equivalence, hence the induced map $H^*(|\mathcal{N}(\mathcal{V}^t)|) \leftarrow H^*(|\mathcal{N}(\mathcal{U}^t)|)$ is an isomorphism. We conclude from the last diagram that $(g^t)^*$ is an isomorphism. $\square$

3.4 A sketch of algorithm

Suppose that we are given a finite set $X \subset E = \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Choose $d \in [1, n-1]$ and $\gamma > 0$. Consider the Čech bundle filtration of dimension d of X. Let $T = \left[0, t_\gamma^{\max}(X)\right)$, $t \in T$ and $i \in [1, d]$. From the previous discussion we can infer an algorithm to solve the following problem:

Compute the persistent Stiefel-Whitney class $w_i^t(X)$ of the Čech bundle filtration of X, using a cohomology computation software.

Denote:

- $\mathbb{X} = (X^t)_{t \geq 0}$ the Čech set filtration of X,
- $\mathbb{S}$ the Čech simplicial filtration of X, and $g^t\colon |S^t| \to X^t$ the shadow map,
- L a triangulation of $\mathcal{G}_d(\mathbb{R}^n)$ and $h\colon \mathcal{G}_d(\mathbb{R}^n) \to |L|$ a homeomorphism,
- $(\mathbb{X}, \mathbb{p})$ the Čech bundle filtration of X,
- $(\mathbb{V}, \mathbb{v})$ the persistent cohomology module of $\mathbb{X}$,
- $w_i \in H^i(\mathcal{G}_d(\mathbb{R}^n))$ the ith Stiefel-Whitney class of the Grassmannian.

Let $t \in T$ and consider the map q^t, as defined in Equation 10:

$$\begin{tikzcd}
{|S^t|} \arrow[r, "g^t"'] \arrow[rrr, bend left, "q^t"] & X^t \arrow[r, "p^t"'] & \mathcal{G}_d(\mathbb{R}^m) \arrow[r, "h"'] & {|L|} .
\end{tikzcd}$$

We propose the following algorithm:

- Subdivise barycentrically S^t until q^t satisfies the star condition. Denote k the number of subdivisions needed.
- Consider a simplicial approximation $f^t\colon \mathrm{sub}^k(S^t) \to L$ to q^t.
- Compute the class $(f^t)^*(w_i)$.

The output $(f^t)^*(w_i)$ is equal to the persistent Stiefel-Whitney class $w_i^t(X)$ at time t, seen in the simplicial cohomology group $H^i(S^t) = H^i(\mathrm{sub}^k(S^t))$. In the following section, we gather some technical details needed to implement this algorithm in practice.

Note that this also gives a way to compute the lifebar of $w_i(X)$. This bar is determined by the value $t^\dagger = \inf\{t \in T, w_i(X) \neq 0\}$. This quantity can be approximated by computing the classes $w_i^t(X)$ for several values of t. We point out that, in order to compute the value $t^\dagger$, there may exist a better algorithm than evaluating the class $w_i^t(X)$ several times.

4 An algorithm when $d = 1$

Even though the last sections described a theoretical way to compute the persistent Stiefel-Whitney classes, some concrete issues are still to be discussed:

- verifying that the star condition is satisfied,
- the Grassmann manifold has to be triangulated,
- in practice, the Vietoris-Rips filtration is prefered to the Čech filtration,
- the parameter γ has to be tuned.

The following subsections will elucidate these points. Concerning the first one, the author is not aware of a computational-explicit process to triangulate the Grassmann manifolds $\mathcal{G}_d(\mathbb{R}^m)$, except when $d = 1$, which corresponds to the projective spaces $\mathcal{G}_1(\mathbb{R}^m)$. We will then restrict to the case $d = 1$.

4.1 The star condition in practice

Let us get back to the context of Subsection 3.2: K, L are two simplicial complexes, K is finite, and $g\colon |K| \to |L|$ is a continuous map. We have seen that finding a simplicial approximation to g reduces to finding a small enough barycentric subdivision $\mathrm{sub}^n(K)$ of K such that $g\colon |\mathrm{sub}^n(K)| \to |L|$ satisfies the star condition, that is, for every vertex v of $\mathrm{sub}^n(K)$, there exists a vertex w of L such that

$$g\left(\left|\overline{\mathrm{St}}\,(v)\right|\right) \subseteq |\mathrm{St}\,(w)|\,.$$

In practice, one can compute the closed star $\overline{\mathrm{St}}\,(v)$ from the finite simplicial complex $\mathrm{sub}^n(K)$. However, computing $g\left(\left|\overline{\mathrm{St}}\,(v)\right|\right)$ requires to evaluate g on the infinite set $\left|\overline{\mathrm{St}}\,(v)\right|$. In order to reduce the problem to a finite number of evaluations of g, we will consider a related property that we call the *weak star condition.*

Definition 4.1 (Weak star condition)**.** A map $g\colon |K| \to |L|$ between topological realizations of simplicial complexes K and L satisfies the *weak star condition* if for every vertex v of $\mathrm{sub}^n(K)$, there exists a vertex w of L such that

$$\left|g\left(\overline{\mathrm{St}}\,(v)^0\right)\right| \subseteq |\mathrm{St}\,(w)|\,,$$

where $\overline{\mathrm{St}}\,(v)^0$ denotes the 0-skeleton of $\overline{\mathrm{St}}\,(v)$, i.e. its vertices.

Observe that the practical verification of the condition $\left|g\left(\overline{\mathrm{St}}\,(v)^0\right)\right| \subseteq |\mathrm{St}\,(w)|$ requires only a finite number of computations. Indeed, one just has to check whether every neighbor v' of v in the graph K^1, v included, satisfies $g(v') \in |\mathrm{St}\,(w)|$. The following lemma rephrases this condition by using the face map $\mathcal{F}_L\colon |L| \to L$ defined in Subsection 3.1. We remind the reader that the face map is defined by the relation $x \in \mathcal{F}_L(x)$ for all $x \in |L|$.

Lemma 4.1. *The map g satisfies the weak star condition if and only if for every vertex v of K, there exists a vertex w of L such that for every neighbor v' of v in K^1, we have*

$$w \in \mathcal{F}_L(g(v')).$$

Proof. Let us show that the assertion "$w \in \mathcal{F}_L(g(v'))$" is equivalent to "$g(v') \in |\mathrm{St}\,(w)|$". Remind that the open star $\mathrm{St}\,(w)$ consists of simplices of L that contain w. Moreover, the topological realization $|\mathrm{St}\,(w)|$ is the union of $|\sigma|$ for $\sigma \in \mathrm{St}\,(w)$. As a consequence, $g(v')$ belongs to $|\mathrm{St}\,(w)|$ if and only if it belongs to $|\sigma|$ for some simplex $\sigma \in L$ that contains w. Equivalently, the face map $\mathcal{F}_L(g(v'))$ contains w. □

Suppose that g satisfies the weak star condition. Let $f\colon K^0 \to L^0$ be a map between vertex sets such that for every $v \in K^0$,

$$\left| g\left(\overline{\mathrm{St}}\,(v)^0 \right) \right| \subseteq \left| \mathrm{St}\,(f(v)) \right|.$$

According to the proof of Lemma 4.1, an equivalent formulation of this condition is: for all neighbor v' of v in K^1,

$$f(v) \in \mathcal{F}_L(g(v')). \tag{11}$$

Such a map is called a *weak simplicial approximation to g*. It plays a similar role as the simplicial approximations to g.

Lemma 4.2. *If $f\colon K^0 \to L^0$ is a weak simplicial approximation to $g\colon |K| \to |L|$, then f is a simplicial map.*

Proof. Let $\sigma = [v_0, ..., v_n]$ be a simplex of K. We have to show that $f(\sigma) = [f(v_0), ..., f(v_n)]$ is a simplex of L. Note that each closed star $\overline{\mathrm{St}}\,(v_i)$ contains σ. Therefore each $\left| g\left(\overline{\mathrm{St}}\,(v_i)^0 \right) \right|$ contains $\left| g\left(\sigma^0\right) \right| = \{g(v_0), ..., g(v_n)\}$. Using the weak simplicial approximation property of f, we deduce that each $|\mathrm{St}\,(f(v_i))|$ contains $\{g(v_0), ..., g(v_n)\}$. Using Lemma 4.3 stated below, we obtain that $[f(v_0), ..., f(v_n)]$ is a simplex of L. □

Lemma 4.3 ([Hat02, Lemma 2C.2])**.** *Let $w_0, ...w_n$ be vertices of a simplicial complex L. Then $\bigcap_{i=0}^n \mathrm{St}\,(w_i) \neq \emptyset$ if and only if $[w_0, ..., w_n]$ is a simplex of L.*

As one can see from the definitions, the weak star condition is weaker than the star condition. Consequently, the simplicial approximation theorem admits the following corollary.

Corollary 4.4. *Consider two simplicial complexes K, L with K finite, and let $g\colon |K| \to |L|$ be a continuous map. Then there exists $n \geq 0$ such that $g\colon |\mathrm{sub}^n(K)| \to |L|$ satisfies the weak star condition.*

However, some weak simplicial approximations to g may not be simplicial approximations, and may not even be homotopic to g. Figure 24 gives such an example.

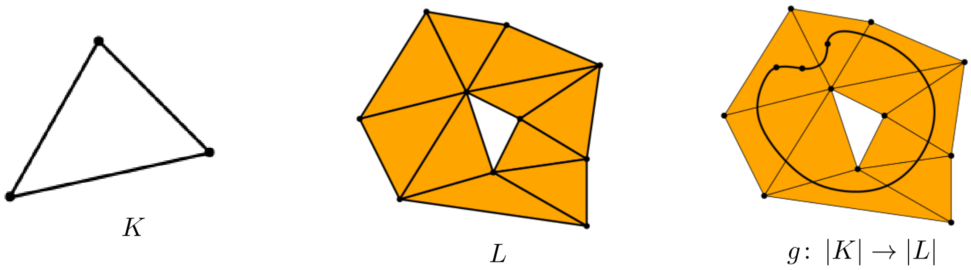


Figure 24: The map g admits a weak simplicial approximation which is constant.

Fortunately, these two notions coincides under the star condition assumption:

Proposition 4.5. *Suppose that g satisfies the star condition. Then every weak simplicial approximation to g is a simplicial approximation.*

Proof. Let f be a weak simplicial approximation to g, and f' any simplicial approximation. Let us show that f and f' are contiguous simplicial maps. Let $\sigma = [v_0, ..., v_n]$ be a simplex of K. We have to show that $[f(v_0), ..., f(v_n), f'(v_0), ..., f'(v_n)]$ is a simplex of L. As we have seen in the proof of Lemma 4.2, each $\left| g\left(\overline{\mathrm{St}}\,(v_i)^0 \right) \right|$ contains $\{g(v_0), ..., g(v_n)\}$. Therefore, by definition of weak simplicial approximations and simplicial approximations, each $|\mathrm{St}\,(f(v_i))|$ and $|\mathrm{St}\,(f'(v_i))|$ contains $\{g(v_0), ..., g(v_n)\}$. We conclude by applying Lemma 4.3. □

Remark that the proof of this proposition can be adapted to obtain the following fact: any two weak simplicial approximations are equivalent—as well as any two simplicial approximations.

Let us comment Proposition 4.5. If K is subdivised enough, then every weak simplicial approximation to g is homotopic to g. But in practice, the number of subdivisions needed by the star condition is not known. We propose to subdivise the complex K until it satisfies the weak star condition, and then use a weak simplicial approximation to g. However, such a weak simplicial approximation may not be homotopic to g, and our algorithm would output a wrong result.

To close this subsection, we state a lemma that gives a quantitative idea of the number of subdivisions needed by the star condition. We say that a Lebesgue number for an open cover $\mathcal{U}$ of a compact metric space X is a positive number ϵ such that every subset of X with diameter less than ϵ is included in some member of the cover $\mathcal{U}$.

Lemma 4.6. *Let $|K|, |L|$ be endowed with metrics. Suppose that $g\colon |K| \to |L|$ is l-Lipschitz with respect to these metrics. Let ϵ be a Lebesgue number for the open cover $\{|\mathrm{St}\,(w)|, w \in L\}$ of $|L|$. Let p be the dimension of K and D an upper bound on the diameter of its faces. Then for $n > \log(\frac{Dl}{\epsilon}) / \log(\frac{p+1}{p})$, the map $g\colon |\mathrm{sub}^n\,(K)| \to |L|$ satisfies the star condition.*

Proof. The map g satisfies the star condition if for every vertex v of K, there exists a vertex w of L such that $g(|\overline{\mathrm{St}}\,(v)|) \subseteq |\mathrm{St}\,(w)|$. Since the cover $\{|\mathrm{St}\,(w)|, w \in L\}$ admits ϵ as a Lebesgue number, it is enough for v to satisfy the following inequality:

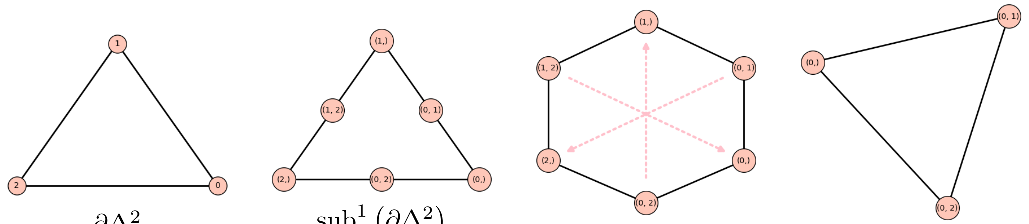
$$\mathrm{diam}\left(g(|\overline{\mathrm{St}}\,(v)|)\right) < \epsilon. \tag{12}$$

Since g is l-Lipschitz, we have $\mathrm{diam}\left(g\left(|\overline{\mathrm{St}}\,(v)|\right)\right) \leq l \cdot \mathrm{diam}\left(|\overline{\mathrm{St}}\,(v)|\right)$. Using the hypothesis $\mathrm{diam}\left(|\overline{\mathrm{St}}\,(v)|\right) \leq D$, Equation 12 leads to the condition $Dl < \epsilon$. Now, we use the fact that a barycentric subdivision reduces the diameter of each face by a factor $\frac{p}{p+1}$. After n barycentric subdivision, the last inequality rewrites $\left(\frac{p}{p+1}\right)^n Dl < \epsilon$. It admits $n > \log(\frac{Dl}{\epsilon}) / \log(\frac{p+1}{p})$ as a solution. □

4.2 Triangulating the projective spaces

As we described in Subsection 4.1, the algorithm we propose rests on a triangulation L of the Grassmannian $\mathcal{G}_1(\mathbb{R}^m)$, together the map $\mathcal{F}_L \circ h\colon \mathcal{G}_1(\mathbb{R}^m) \to L$, where $h\colon \mathcal{G}_1(\mathbb{R}^m) \to |L|$ is a homeomorphism and $\mathcal{F}_L\colon \mathcal{G}_1(\mathbb{R}^m) \to L$ is the face map. In the following, we also refer to $\mathcal{F} := \mathcal{F}_L \circ h$ as the face map.

We will use the following folklore triangulation of the projective space $\mathcal{G}_1(\mathbb{R}^m)$. It uses the fact that the quotient of the sphere $\mathbb{S}_{m-1}$ by the antipodal relation gives $\mathcal{G}_1(\mathbb{R}^m)$. Let Δ^m denote the standard m-simplex, $v_0, ..., v_m$ its vertices, and $\partial\Delta^m$ its boundary. The simplicial complex $\partial\Delta^m$ is a triangulation of the sphere $\mathbb{S}_{m-1}$. Denote its first barycentric subdivision as $\mathrm{sub}^1\,(\partial\Delta^m)$. The vertices of $\mathrm{sub}^1\,(\partial\Delta^m)$ are in bijection with the non-empty proper subsets of $\{v_0, ..., v_m\}$ (see Subsection 3.2). Consider the equivalence relation on these vertices which associates a vertex to its complement. The quotient simplicial complex under this relation, L, is a triangulation of $\mathcal{G}_1(\mathbb{R}^m)$.



$\partial\Delta^2$ | $\mathrm{sub}^1\,(\partial\Delta^2)$ | Equivalence relation | Quotient complex L

Figure 25: Triangulating $\mathcal{G}_1(\mathbb{R}^2)$.

Let us now describe how to define the homeomorphism $h\colon \mathcal{G}_1(\mathbb{R}^m) \to |L|$. First, embed Δ^m in $\mathbb{R}^{m+1}$ via $v_i \mapsto (0, ..., 0, 1, 0, ...)$, where 1 sits at the ith coordinate. Its image lies on a m-dimensional affine subspace P, with origin being the barycenter of $v_0, ..., v_m$. Seen in P, the vertices of Δ^m now belong to the sphere centered at the origin and of radius $\sqrt{\frac{m}{m+1}}$ (see Figure 26). Let us denote this sphere as $\mathbb{S}_{m-1}$. Next, subdivise barycentrically $\partial\Delta^m$ once, and project each vertex of $\mathrm{sub}^1(\partial\Delta^m)$ on $\mathbb{S}_{m-1}$. By taking the convex hulls of its faces, we now see $\left|\mathrm{sub}^1(\partial\Delta^m)\right|$ as a subset of P. Define an application $p\colon \mathbb{S}_{m-1} \to \left|\mathrm{sub}^1(\partial\Delta^m)\right|$ as follows: for every $x \in \mathbb{S}_{m-1}$, the image $p(x)$ is the unique intersection point between the segment $[0, x]$ and the set $\left|\mathrm{sub}^1(\partial\Delta^m)\right|$. The application p can also be seen as the inverse function of the projection on $\mathbb{S}_{m-1}$, written $\mathrm{proj}_{\mathbb{S}_{m-1}}(\cdot)\colon \left|\mathrm{sub}^1(\partial\Delta^m)\right| \to \mathbb{S}_{m-1}$.

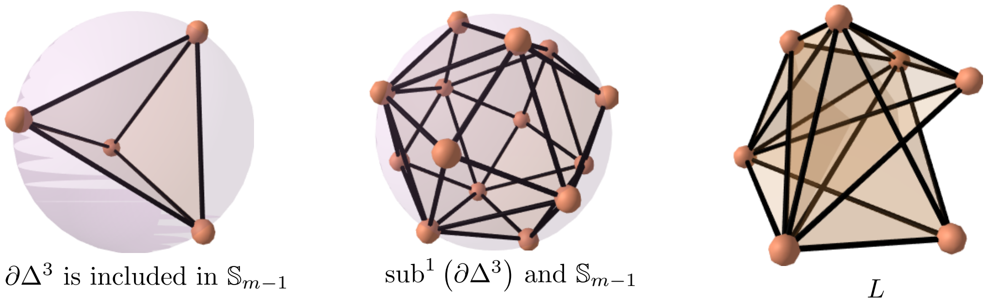


Figure 26: Triangulating $\mathcal{G}_1(\mathbb{R}^3)$.

The next lemma shows that the antipodal relation on $\mathbb{S}_{m-1}$ can be pulled-back to $\left|\mathrm{sub}^1(\partial\Delta^m)\right|$ via p, and it corresponds to the equivalence relation we defined on $\mathrm{sub}^1(\partial\Delta^m)$. As a consequence, we can factorize $p\colon \mathbb{S}_{m-1} \to \left|\mathrm{sub}^1(\partial\Delta^m)\right|$ as

$$h\colon \left(\mathbb{S}_{m-1}/\sim\right) \to \left(\left|\mathrm{sub}^1(\partial\Delta^m)\right|/\sim\right),$$

and we can identify these spaces with

$$h\colon \mathcal{G}_1(\mathbb{R}^m) \to |L|,$$

giving the desired homeomorphism.

Lemma 4.7. *For any vertex $x \in \mathrm{sub}^1(\partial\Delta^m)$, denote by $|x|$ its embedding in P. Let $-|x|$ denote the image of $|x|$ by the antipodal relation on $\mathbb{S}_{m-1}$. Denote by y the image of x by the relation on $\mathrm{sub}^1(\partial\Delta^m)$. Then $y = -|x|$.*

More generally, pulling back the antipodal relation onto $\left|\mathrm{sub}^1(\partial\Delta^m)\right|$ via p gives the relation we defined on $\mathrm{sub}^1(\partial\Delta^m)$.

Proof. Pick a vertex x of $\mathrm{sub}^1(\partial\Delta^m)$. It can be described as a proper subset $\{v_i, i \in I\}$ of the vertex set $(\partial\Delta^m)^0 = \{v_0, ..., v_m\}$, where $I \subset [0, m]$. According to the relation on $(\partial\Delta^m)$, the vertex x is in relation with the vertex y described by the proper subset $\{v_i, i \in I^c\}$.

The point x can be written in barycentric coordinates as $\frac{1}{\mathrm{card}(I)}\sum_{i\in I}|v_i|$. Seen in P, $|x|$ can be written $|x| = \mathrm{proj}_{\mathbb{S}_{m-1}}\left(\sum_{i\in I} v_i\right)$. Similarly, $|y|$ can be written $|y| = \mathrm{proj}_{\mathbb{S}_{m-1}}\left(\sum_{i\in I^c} v_i\right)$.

Now, denote by 0 the origin of the hyperplane P, and embed the vertices $v_0, ..., v_m$ in P. Observe that

$$0 = \sum_{i\leq 0} v_i = \sum_{i\in I} v_i + \sum_{i\in I^c} v_i.$$

Hence $-\sum_{i\in I} v_i = \sum_{i\in I^c} v_i$, and we deduce that

$$-|x| = \mathrm{proj}_{\mathbb{S}_{m-1}}\left(-\sum_{i\in I} v_i\right) = \mathrm{proj}_{\mathbb{S}_{m-1}}\left(\sum_{i\in I^c} v_i\right) = |y|.$$

Applying the same reasoning, one obtains the following result: for every simplex σ of $\mathrm{sub}^1(\partial\Delta^m)$, if ν denotes the image of σ by the relation of $\mathrm{sub}^1(\partial\Delta^m)$, then the image of $|\sigma|$ by the antipodal relation is also $|\nu|$. As a consequence, these two relations coincide. $\square$

At a computational level, let us describe how to compute the face map $\mathcal{F}\colon \mathcal{G}_1(\mathbb{R}^m) \to L$. Since $\mathcal{F}$ can be obtained as a quotient, it is enough to compute the face map of the sphere, $\mathcal{F}'\colon \mathbb{S}_{m-1} \to \mathrm{sub}^1(\partial\Delta^m)$, which corresponds to the homeomorphism $p\colon \mathbb{S}_{m-1} \to \left|\mathrm{sub}^1(\partial\Delta^m)\right|$. It is given by the following lemma, which can be used in practice.

Lemma 4.8. *For every $x \in \mathbb{S}_{m-1}$, the image of x by the face map $\mathcal{F}'$ is equal to the intersection of all maximal faces $\sigma = [w_0, ..., w_m]$ of $\mathrm{sub}^1(\partial\Delta^m)$ that satisfies the following conditions: denoting by x_0 any point of the affine hyperplane spanned by $\{w_0, ..., w_m\}$, and by h a vector orthogonal to the corresponding linear hyperplane,*

- *the inner product $\langle x, h\rangle$ has the same sign as $\langle x_0, h\rangle$,*
- *the point $\frac{\langle x_0,h\rangle}{\langle x,h\rangle}x$, which is included in the affine hyperplane spanned by $\{w_0, ..., w_m\}$, has nonnegative barycentric coordinates.*

Proof. Recall that for every $x \in \mathbb{S}_{m-1}$, the image $p(x)$ is defined as the unique intersection point between the segment $[0, x]$ and the set $\left|\mathrm{sub}^1(\partial\Delta^m)\right|$. Besides, the face map $\mathcal{F}'(x)$ is the unique simplex $\sigma \in \mathrm{sub}^1(\partial\Delta^m)$ such that $p(x) \in |\sigma|$. Equivalently, $\mathcal{F}'(x)$ is equal to the intersection of all maximal faces $\sigma \in \mathrm{sub}^1(\partial\Delta^m)$ such that $p(x)$ belongs to the closure $\overline{|\sigma|}$.

Consider any maximal face $\sigma = [w_0, ..., w_m]$ of $\mathrm{sub}^1(\partial\Delta^m)$. The first condition of the lemma ensures that the segment $[0, x]$ intersects the affine hyperplane spanned by $\{w_0, ..., w_m\}$. In this case, a computation shows that this intersection consists of the point $\frac{\langle x_0,h\rangle}{\langle x,h\rangle}x$. Then, the second condition of the lemma tests whether this point belongs to the convex hull of $\{w_0, ..., w_k\}$. In conclusion, if σ satisfies these two conditions, then $p(x) \in \overline{|\sigma|}$. $\square$

As a remark, let us point out that the verification of the conditions of this lemma is subject to numerical errors. In particular, the point $\frac{\langle x_0,h\rangle}{\langle x,h\rangle}x$ may have nonnegative coordinates, yet mathematical softwares may return (small) negative values. Consequently, the algorithm may recognize less maximal faces that satisfy these conditions, hence return a simplex that strictly contains the wanted simplex $\mathcal{F}'(x)$. Nonetheless, such an error will not affect the output of the algorithm. Indeed, if we denote by $\widetilde{\mathcal{F}'}$ the face map computed by the algorithm, we have that $\mathcal{F}'(x) \subseteq \widetilde{\mathcal{F}'}(x)$ for all $x \in \mathbb{S}_{m-1}$. As a consequence of Lemma 4.1, $\widetilde{\mathcal{F}'}$ satisfies the weak star condition if $\mathcal{F}'$ does, and Equation 11 shows that every weak simplicial approximations for $\mathcal{F}'$ are weak simplicial approximations for $\widetilde{\mathcal{F}'}$. Since every weak simplicial approximations are homotopic, we obtain that the induced maps in cohomology are equal, therefore the output of the algorithm is unchanged.

4.3 Vietoris-Rips version of the Čech bundle filtration

We still consider a subset $X \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. Denote by $\mathbb{X}$ the corresponding Čech set filtration, and by $\mathbb{S} = (S^t)_{t\geq 0}$ the simplicial Čech filtration. For every $t \geq 0$, let R^t be the flag complex of S^t, i.e. the clique complex of the 1-skeleton $(S^t)^1$ of S^t. It is known as the *Vietoris-Rips complex* of X at time t. The collection $\mathbb{R} = (R^t)_{t\geq 0}$ is called the *Vietoris-Rips filtration* of X. The simplicial filtrations $\mathbb{S}$ and $\mathbb{R}$ are multiplicatively $\sqrt{2}$-interleaved [BLM+17, Theorem 3.1]. In other words, for every $t \geq 0$, we have

$$S^t \subseteq R^t \subseteq S^{\sqrt{2}t}.$$

Let $\gamma > 0$ and consider the Čech bundle filtration $(\mathbb{X}, \mathbb{p})$ of X. Suppose that its maximal filtration value $t_\gamma^{\max}(X)$ is positive. Let $|\mathbb{R}| = (|R^t|)_{t\geq 0}$ denote the topological realization of the

Vietoris-Rips filtration. We can give $|\mathbb{R}|$ a vector bundle filtration structure with $(p')^t \colon |R^t| \to \mathcal{G}_d(\mathbb{R}^m)$ defined as

$$(p')^t = p^{\sqrt{2}t} \circ i^t,$$

where $p^{\sqrt{2}t}$ denotes the maps of the Čech bundle filtration $(\mathbb{X}, \mathbb{p})$, and i^t denotes the inclusion $|R^t| \hookrightarrow \left|S^{\sqrt{2}t}\right|$. These maps fit in the following diagram:

$$\begin{tikzcd}
|R^t| \arrow[r, hook, "i^t"] \arrow[drr, dashed, "(p')^t"'] & \left|S^{\sqrt{2}t}\right| \arrow[r, phantom, "="] & X^{\sqrt{2}t} \arrow[d, "p^{\sqrt{2}t}"] \\
& & \mathcal{G}_d(\mathbb{R}^m)
\end{tikzcd}$$

This new vector bundle filtration is defined on the index set $T' = \left[0, \frac{1}{\sqrt{2}} t_\gamma^{\max}(X)\right)$.

It is clear from the construction that the vector bundle filtrations $(\mathbb{X}, \mathbb{p})$ and $(|\mathbb{R}|, \mathbb{p}')$ are multiplicatively $\sqrt{2}$-interleaved, with an interleaving that preserves the persistent Stiefel-Whitney classes. This property is a multiplicative equivalent of Theorem 2.3.

We recall the reader that, if X is a subset of $\mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m)$, then the maximal filtration value of the Čech bundle filtration on X is $t_\gamma^{\max}(X) = \frac{\sqrt{2}}{2}\gamma$ (see Equation 5). Consequently, the maximal filtration value of its Vietoris-Rips version is $\frac{1}{2}\gamma$.

From an application perspective, we choose to work with the Vietoris-Rips filtration since it is easier to compute. Indeed, its construction only relies on computing pairwise distances and finding cliques in graphs.

4.4 Choice of the parameter γ

This subsection is devoted to discussing the influence of the parameter $\gamma > 0$. Recall that γ affects the norm $\|\cdot\|_\gamma$ we chose on $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$:

$$\|(x, A)\|_\gamma^2 = \|x\|^2 + \gamma^2 \|A\|_{\mathrm{F}}^2.$$

Let $X \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$. If $\gamma_1 \leq \gamma_2$ are two positive real numbers, the corresponding Čech filtrations $\mathbb{X}_1$ and $\mathbb{X}_2$, as well as the Čech bundle filtrations $(\mathbb{X}_1, \mathbb{p}_1)$ and $(\mathbb{X}_2, \mathbb{p}_2)$, are $\frac{\gamma_2}{\gamma_1}$-interleaved multiplicatively. This comes from the straightforward inequality

$$\|\cdot\|_{\gamma_1} \ \leq \ \|\cdot\|_{\gamma_2} \ \leq \ \frac{\gamma_2}{\gamma_1}\|\cdot\|_{\gamma_1}.$$

Note that we also have the additive inequality

$$\|(x, A)\|_{\gamma_1} \ \leq \ \|(x, A)\|_{\gamma_2} \ \leq \ \|(x, A)\|_{\gamma_1} + \sqrt{\gamma_2^2 - \gamma_1^2}\|A\|_{\mathrm{F}}.$$

One deduces that the Čech bundle filtrations $(\mathbb{X}_1, \mathbb{p}_1)$ and $(\mathbb{X}_2, \mathbb{p}_2)$ are $\sqrt{\gamma_2^2 - \gamma_1^2} \cdot t^{\max}(X)$-interleaved additively, where $t^{\max}(X)$ is the maximal filtration value when $\gamma = 1$. As a consequence of these interleavings, when the values γ_1 and γ_2 are close, the persistence diagrams and the lifebars of the persistent Stiefel-Whitney classes are close (see Theorem 2.3).

As a general principle, one would choose the parameter γ to be large, since it would lead to large filtrations. More precisely, if $t_{\gamma_1}^{\max}(X)$ and $t_{\gamma_2}^{\max}(X)$ denote repectively the maximal filtration values of $(\mathbb{X}_1, \mathbb{p}_1)$ and $(\mathbb{X}_2, \mathbb{p}_2)$, then $t_{\gamma_1}^{\max}(X) = \gamma_1 \cdot t^{\max}(X)$ and $t_{\gamma_2}^{\max}(X) = \gamma_2 \cdot t^{\max}(X)$, as in Equation 4. Moreover, we have the following inclusion:

$$X_1^{t_{\gamma_1}^{\max}(X)} \subseteq X_2^{t_{\gamma_2}^{\max}(X)},$$

where $X_1^{t_{\gamma_1}^{\max}(X)}$ denotes the thickening of X with respect to the norm $\|\cdot\|_{\gamma_1}$, and $X_2^{t_{\gamma_2}^{\max}(X)}$ with respect to $\|\cdot\|_{\gamma_2}$. This inclusion can be proven from the following fact, valid for every $x \in \mathbb{R}^n$ and $A \in \mathcal{M}(\mathbb{R}^m)$ such that $\|A\|_{\mathrm{F}} \leq t^{\max}(X)$:

$$\|(x, A)\|_{\gamma_1} \leq t_{\gamma_1}^{\max}(X) \implies \|(x, A)\|_{\gamma_2} \leq t_{\gamma_2}^{\max}(X).$$

Hence larger parameters γ lead to larger maximal filtration values and larger filtrations.

However, as we show in the following examples, different values of γ may result in different behaviours of the persistent Stiefel-Whitney classes. In Example 4.10, large values of γ highlight properties of the dataset that are not consistent with the underlying vector bundle: the persistent Stiefel-Whitney class is nonzero, yet the vector bundle is orientable. Notice that, so far, we always picked the value $\gamma = 1$, for it seemed experimentally relevant with the datasets we chose.

Example 4.9. Consider the set $Y \subset \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ representing the Mobius band, as in Example 2.2 of Subsection 2.2:

$$Y = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\frac{\theta}{2})^2 & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin(\frac{\theta}{2})^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

As we show in Appendix B.1, Y is a circle, included in a 2-dimensional affine subspace of $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$. Its radius is $\sqrt{1 + \frac{\gamma^2}{2}}$. As a consequence, the persistence of the Čech filtration of Y consists of two bars: one H^0-feature, the bar $[0, +\infty)$, and one H^1-feature, the bar $\left[0, \sqrt{1 + \frac{\gamma^2}{2}}\right)$.

For any $\gamma > 0$, the maximal filtration value of the Čech bundle filtration of Y is $t_\gamma^{\max}(Y) = \frac{\sqrt{2}}{2}\gamma$. Moreover, the persistent Stiefel-Whitney class $w_1^t(Y)$ is nonzero all along the filtration.

In this example, we see that the parameter γ does not influence the qualitative interpretation of the persistent Stiefel-Whitney class. It is always nonzero where it is defined. The following example shows a case where γ does influence the persistent Stiefel-Whitney class.

Example 4.10. Consider the set $X \subset \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ representing the normal bundle of the circle $\mathbb{S}_1$, as in Example 2.2:

$$X = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

As we show in Appendix B.2, X is a subset of a 2-dimensional flat torus embedded in $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$, hence can be seen as a torus knot.

Before studying the Čech bundle filtration of X, we discuss the Čech filtration $\mathbb{X}$. Its behaviour depends on γ:

- if $\gamma \leq \frac{\sqrt{2}}{2}$, then X^t retracts on a circle for $t \in [0, 1)$, X^t retracts on a 3-sphere for $t \in \left[1, \sqrt{1 + \frac{1}{2}\gamma^2}\right)$, and X^t retracts on a point for $t \geq \sqrt{1 + \frac{1}{2}\gamma^2}$.
- if $\gamma \geq \frac{\sqrt{2}}{2}$, then X^t retracts on a circle for $t \in [0, 1)$, X^t retracts on another circle for $t \in \left[1, \frac{\sqrt{2}}{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}\right)$, X^t retracts on a 3-sphere for $t \in \left[\frac{\sqrt{2}}{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}, \sqrt{1 + \frac{1}{2}\gamma^2}\right)$, and X^t has the homotopy type of a point for $t \geq \sqrt{1 + \frac{1}{2}\gamma^2}$.

Let us interpret these facts. If $\gamma \leq \frac{\sqrt{2}}{2}$, then the persistent cohomology of X looks similar to the persistent cohomology of the underlying set $\left\{ \left(\begin{smallmatrix} \cos(\theta) \\ \sin(\theta) \end{smallmatrix} \right), \theta \in [0, 2\pi) \right\} \subset \mathbb{R}^2$, but with a H^3 cohomology feature added. Besides, if $\gamma \geq \frac{\sqrt{2}}{2}$, a new topological feature appears in the H^1-barcode: the bar $\left[1, \sqrt{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}\right)$. These barcodes are depicted in Figures 27 and 28.

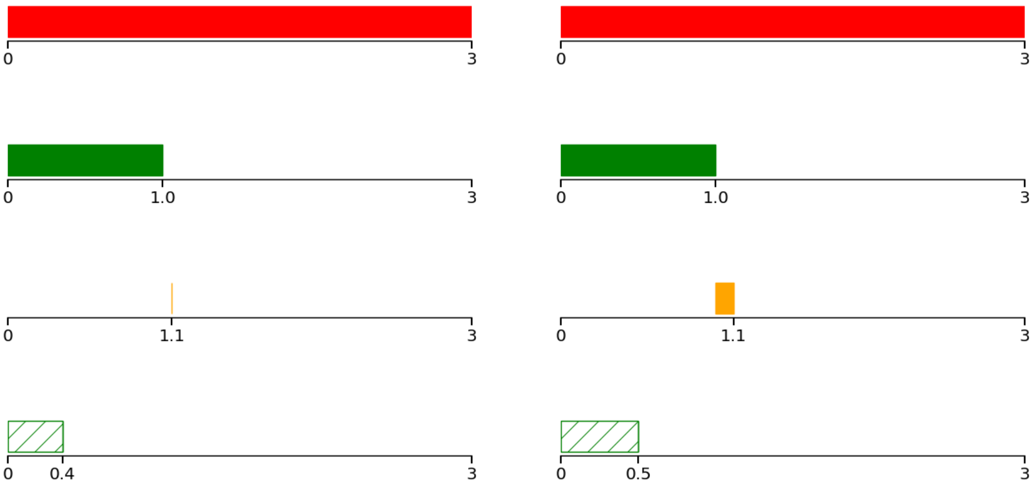


Figure 27: H^0-, H^1-, H^3-barcodes and lifebar of the first persistent Stiefel-Whitney class of X with $\gamma = \frac{1}{2}$ (left) and $\gamma = \frac{\sqrt{2}}{2}$ (right).

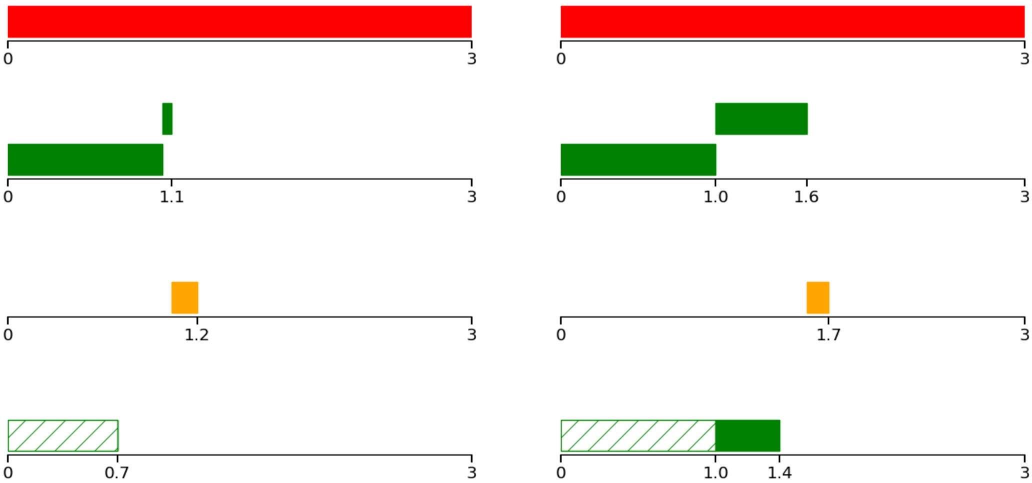


Figure 28: H^0-, H^1-, H^3-barcodes and lifebar of the first persistent Stiefel-Whitney class of X with $\gamma = 1$ (left) and $\gamma = 2$ (right).

Let us now discuss the corresponding Čech bundle filtrations. For any $\gamma > 0$, the maximal filtration value of the Čech bundle filtration of X is $t_\gamma^{\max}(X) = \frac{\sqrt{2}}{2}\gamma$. We observe two behaviours:

- if $\gamma \leq \frac{\sqrt{2}}{2}$, then $w_1^t(X)$ is zero all along the filtration,
- if $\gamma > \frac{\sqrt{2}}{2}$, then $w_1^t(X)$ is nonzero from $t^\dagger = 1$.

This in proven in Appendix B.2. To conclude, this persistent Stiefel-Whitney class is consistent with the underlying bunlde—the normal bundle of the circle, which is trivial—only for $t \leq 1$.

5 Conclusion

In this paper we defined the persistent Stiefel-Whitney classes of vector bundle filtrations. We proved that they are stable with respect to the interleaving distance between vector bundle filtrations. We studied the particular case of Čech bundle filtrations of subsets of $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, and showed that they yield consistent estimators of the usual Stiefel-Whitney classes of some underlying vector bundle. Moreover, when the dimension of the bundle is 1 and X is finite, we proposed an algorithm to compute the persistent Stiefel-Whitney classes.

Our algorithm is limited to the bundles of dimension 1, since we only implemented triangulations of the Grassmannian $\mathcal{G}_d(\mathbb{R}^m)$ when $d = 1$. However, any other triangulation of $\mathcal{G}_d(\mathbb{R}^m)$, with a computable face map, could be included in the algorithm without any modification. We also described a way to compute the lifebar of the persistent Stiefel-Whitney classes, by evaluating the class for several values of t.

A Supplementary material for Section 2

We prove Lemma 2.1, stated page 12.

Proof of Lemma 2.1. Note that $\mathcal{G}_d(\mathbb{R}^m)$ is contained in the linear subspace $\mathcal{S}$ of symmetric matrices. Therefore, to project a matrix $A \in \mathcal{M}(\mathbb{R}^m)$ onto $\mathcal{G}_d(\mathbb{R}^m)$, we may project on $\mathcal{S}$ first. It is well known that the projection of A onto $\mathcal{S}$ is the matrix $A^s = \frac{1}{2}(A + {}^tA)$.

Suppose now that we are given a symmetric matrix B. Let it be diagonalized as $B = OD^tO$. A projection of B onto $\mathcal{G}_d(\mathbb{R}^m)$ is a matrix P which minimizes the following quantity:

$$\min_{P \in \mathcal{G}_d(E)} \|B - P\|_{\mathrm{F}}. \tag{13}$$

This problem is equivalent to

$$\min_{P \in \mathcal{G}_d(E)} \|D - P\|_{\mathrm{F}}$$

via $P \mapsto {}^tOPO$. Now, let $e_1, \cdots, e_n$ denote the canonical basis of $\mathbb{R}^m$. We have

$$\begin{aligned} \|D - P\|_{\mathrm{F}}^2 &= \|D\|_{\mathrm{F}}^2 + \|P\|_{\mathrm{F}}^2 - 2\langle D, P\rangle_{\mathrm{F}} \\ &= \|D\|_{\mathrm{F}}^2 + \|P\|_{\mathrm{F}}^2 - 2\sum \langle \lambda_i e_i, P(e_i)\rangle, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathrm{F}}$ is the Frobenius inner product, and $\langle\cdot,\cdot\rangle$ the usual inner product on $\mathbb{R}^m$. Therefore, Equation 13 is a problem equivalent to

$$\max_{P \in \mathcal{G}_d(E)} \sum \lambda_i \langle e_i, P(e_i)\rangle.$$

Since P is an orthogonal projection, we have $\langle e_i, P(e_i)\rangle = \langle P(e_i), P(e_i)\rangle = \|P(e_i)\|^2$ for all $i \in [1, n]$. Moreover, $d = \|P\|_{\mathrm{F}}^2 = \sum \|P(e_i)\|^2$. Denoting $p_i = \|P(e_i)\|^2 \in [0, 1]$, we finally obtain the following alternative formulation of Equation 13:

$$\max_{\substack{p_1,\dots p_n \in [0,1] \\ p_1+\dots+p_n = d}} \sum \lambda_i p_i.$$

Using that $\lambda_1 \geq \dots \geq \lambda_n$, we see that this maximum is attained when $p_0 = \dots = p_d = 1$ and $p_{d+1} = \dots = p_n = 0$. Consequently, a minimizer of Equation 13 is $P = J_d$, where J_d is the diagonal matrix whose first d terms are 1, and the other ones are zero. Moreover, it is unique if $\lambda_d \neq \lambda_{d+1}$.

As a consequence of these considerations, we obtain the following characterization: for every $B \in \mathcal{M}(\mathbb{R}^m)$,

$$B \in \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m)) \iff \lambda_d(B^s) = \lambda_{d+1}(B^s). \tag{14}$$

Let us now show that for every matrix $A \in \mathcal{M}(\mathbb{R}^m)$, we have

$$\operatorname{dist}(A, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))) = \frac{\sqrt{2}}{2}\big|\lambda_d(A^s) - \lambda_{d+1}(A^s)\big|.$$

First, remark that

$$\operatorname{dist}(A, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))) = \operatorname{dist}(A^s, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m)))\,. \tag{15}$$

Indeed, if B is a projection of A on $\operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))$, then B^s is still in $\operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))$ according to Equation 14, and

$$\operatorname{dist}(A, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))) = \|A - B\|_{\mathrm{F}} \geq \|A^s - B^s\|_{\mathrm{F}} \geq \operatorname{dist}(A^s, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m)))\,.$$

Conversely, if B is a projection of A^s on $\operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))$, then $\hat{B} = B + A - A^s$ is still in $\operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))$, and

$$\operatorname{dist}(A, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))) \leq \|A - \hat{B}\|_{\mathrm{F}} = \|A^s - B\|_{\mathrm{F}} = \operatorname{dist}(A^s, \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m)))\,.$$

We deduce Equation 15.

Now, let $A \in \mathcal{S}$ and $B \in \operatorname{med}(\mathcal{G}_d(\mathbb{R}^m))$. Let $e_1, ..., e_n$ be a basis of $\mathbb{R}^m$ that diagonalizes A. Writing $\|A - B\|_{\mathrm{F}} = \sum \|A(e_i) - B(e_i)\|^2 = \sum \|\lambda_i(A)e_i - B(e_i)\|^2$, it is clear that the closest matrix B must satisfy $B(e_i) = \lambda_i(B)e_i$, with

- $\lambda_i(B) = \lambda_i(A)$ for $i \notin \{d, d+1\}$,
- $\lambda_d(B) = \lambda_{d+1}(B) = \frac{1}{2}(\lambda_d(A) + \lambda_{d+1}(A))$.

We finally compute

$$\begin{aligned}
\|A - B\|_{\mathrm{F}}^2 &= \sum \|\lambda_i(A)e_i - \lambda_i(B)e_i\|^2 \\
&= |\lambda_d(A) - \lambda_d(B)|^2 + |\lambda_{d+1}(A) - \lambda_{d+1}(B)|^2 \\
&= \frac{1}{2}\,|\lambda_d(A) - \lambda_{d+1}(A)|^2\,.
\end{aligned}$$

□

B Supplementary material for Section 4

B.1 Study of Example 4.9

We consider the set

$$X = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\frac{\theta}{2})^2 & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin(\frac{\theta}{2})^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

To study the Čech filtration of X, we will apply the following affine transformation: let Y be the subset of $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ defined as

$$Y = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \gamma \begin{pmatrix} \cos(\frac{\theta}{2})^2 & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin(\frac{\theta}{2})^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

and let $\mathbb{Y} = (Y^t)_{t \geq 0}$ be the Čech filtration of Y in $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ endowed with the usual norm $\|(x, A)\|_1 = \sqrt{\|x\|^2 + \|A\|_{\mathrm{F}}^2}$. We recall that the Čech filtration of X, denoted $\mathbb{X} = (X^t)_{t \geq 0}$, is defined with respect to the norm $\|\cdot\|_\gamma$. It is clear that, for every $t \geq 0$, the thickenings X^t and Y^t are homeomorphic via the application

$$\begin{aligned}
h \colon \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2) &\longrightarrow \mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2) \\
(x, A) &\longmapsto O + (x, \gamma A).
\end{aligned}$$

As a consequence, the persistence cohomology modules associated to $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic.

Next, notice that Y is a subset of the affine subspace of dimension 2 of $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ with origin O and spanned by the vectors e_1 and e_2, where

$$O = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{\gamma}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \qquad e_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \frac{\gamma}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \qquad e_2 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \frac{\gamma}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Indeed, using the equality

$$\left(\begin{pmatrix} \cos(\frac{\theta}{2})^2 & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin(\frac{\theta}{2})^2 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

we obtain

$$Y = O + \{\cos(\theta)e_1 + \sin(\theta)e_2, \quad \theta \in [0, 2\pi)\}.$$

We see that Y is a circle, of radius $\|e_1\| = \|e_2\| = \sqrt{1 + \frac{\gamma^2}{2}}$.

Let E denotes the affine space with origin O and spanned by the vectors e_1 and e_2. Lemma B.1, stated below, shows that the persistent cohomology of Y, seen in the ambient space $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$, is the same as the persistent cohomology of Y restricted to the subspace E. As a consequence, Y has the same persistence as a circle of radius $\sqrt{1 + \frac{\gamma^2}{2}}$ in the plane. Its barcode can be described as follows:

- one H^0-feature: the bar $[0, +\infty)$,
- one H^1-feature: the bar $\left[0, \sqrt{1 + \frac{\gamma^2}{2}}\right)$.

Lemma B.1. *Let $Y \subset \mathbb{R}^n$ be any subset, and define $\check{Y} = Y \times \{(0, ..., 0)\} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let these spaces be endowed with the usual Euclidean norms. Then the Čech filtrations of Y and $\check{Y}$ yields isomorphic persistence modules.*

Proof. Let $\mathrm{proj}_n \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the projection on the first n coordinates. One verifies that, for every $t \geq 0$, the map $\mathrm{proj}_n \colon \check{Y}^t \to Y^t$ is a homotopy equivalence. At cohomology level, these maps induce an isomorphism of persistence modules. □

Let us now study the Čech bundle filtration of Y, denoted $(\mathbb{Y}, \mathbb{p})$. According to Equation 5, its filtration maximal value is $t^{\max}(Y) = t^{\max}_{\gamma}(X) = \frac{\gamma}{\sqrt{2}}$. Note that $\frac{\gamma}{\sqrt{2}}$ is lower than $\sqrt{1 + \frac{\gamma^2}{2}}$, which is the radius of the circle Y. Hence, for $t < t^{\max}(Y)$, the inclusion $Y \hookrightarrow Y^t$ is a homotopy equivalence. Consider the following commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, "p^0"'] & & Y^t \arrow[dl, "p^t"] \\ & \mathcal{G}_1(\mathbb{R}^2) & \end{tikzcd}$$

It induces the following diagram in cohomology:

$$\begin{tikzcd} H^*(Y) & & H^*(Y^t) \arrow[ll, "\sim"'] \\ & H^*(\mathcal{G}_1(\mathbb{R}^2)) \arrow[ul, "(p^0)^*"] \arrow[ur, "(p^t)^*"'] & \end{tikzcd}$$

The horizontal arrow is an isomorphism. Hence the map $(p^t)^* \colon H^*(Y^t) \leftarrow H^*(\mathcal{G}_1(\mathbb{R}^2))$ is equal to $(p^0)^*$. We only have to understand $(p^0)^*$.

Remark that the map $p^0 \colon Y \to \mathcal{G}_1(\mathbb{R}^2)$ can be seen as the universal bundle of the circle. Therefore $(p^0)^* \colon H^*(Y) \leftarrow H^*(\mathcal{G}_1(\mathbb{R}^m))$ is nontrivial. Alternatively, p^0 can be seen as a map between two circles. It is injective, hence its degree (modulo 2) is one. We still deduce that $(p^0)^*$ is nontrivial. As a consequence, the persistent Stiefel-Whitney class $w_1^t(X)$ is nonzero for every $t < t^{\max}(Y)$.

B.2 Study of Example 4.10

We consider the set

$$X = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

As we explained in the previous subsection, the Čech filtration of X with respect to the norm $\|\cdot\|_\gamma$ yields the same persistence as the Čech filtration of Y with respect to the usual norm $\|\cdot\|$, where

$$Y = \left\{ \left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \gamma \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} \right), \theta \in [0, 2\pi) \right\}.$$

Notice that Y is a subset of the affine subspace of dimension 4 of $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$ with origin $O = \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \frac{1}{2} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right)$ and spanned by the vectors e_1, e_2, e_3 and e_4, where

$$e_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \qquad e_2 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$e_3 = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \qquad e_4 = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Indeed, Y can be written as

$$Y = O + \left\{ \cos(\theta)e_1 + \sin(\theta)e_2 + \frac{\gamma}{\sqrt{2}}\cos(2\theta)e_3 + \frac{\gamma}{\sqrt{2}}\sin(2\theta)e_4, \quad \theta \in [0, 2\pi) \right\}.$$

This comes from the equality

$$\begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Observe that Y is a torus knot, i.e. a simple closed curve included in the torus $\mathbb{T}$, defined as

$$\mathbb{T} = O + \left\{ \cos(\theta)e_1 + \sin(\theta)e_2 + \frac{\gamma}{\sqrt{2}}\cos(\nu)e_3 + \frac{\gamma}{\sqrt{2}}\sin(\nu)e_4, \quad \theta, \nu \in [0, 2\pi) \right\}.$$

The curve Y winds one time around the first circle of the torus, and two times around the second one. It is known as the torus knot $(1, 2)$.

Let E denotes the affine subspace with origin O and spanned by e_1, e_2, e_3, e_4. Since Y is a subset of E, it is equivalent to study the Čech filtration of Y restricted to this subset (as in Lemma B.1). We will denote the coordinates of points $x \in E$ with respect to the orthonormal basis (e_1, e_2, e_3, e_4). That is, a tuple (x_1, x_2, x_3, x_4) will refer to the point $O + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$ of E. Seen in E, the set Y can be written as

$$Y = \left\{ \left(\cos(\theta), \sin(\theta), \frac{\gamma}{\sqrt{2}}\cos(2\theta), \frac{\gamma}{\sqrt{2}}\sin(2\theta) \right), \theta \in [0, 2\pi) \right\}.$$

For every $\theta \in [0, 2\pi)$, we will denote $y_\theta = \left(\cos(\theta), \sin(\theta), \frac{\gamma}{\sqrt{2}}\cos(2\theta), \frac{\gamma}{\sqrt{2}}\sin(2\theta) \right)$.

Figure 29: Representations of the set Y, lying on a torus, for a small value of γ (left) and a large value of γ (right).

We now state two lemmas that will be useful in what follows.

Lemma B.2. *For every $\theta \in [0, 2\pi)$, the map $\theta' \mapsto \|y_\theta - y_{\theta'}\|$ admits the following critical points:*

- $\theta' - \theta = 0$ *and* $\theta' - \theta = \pi$ *if* $\gamma \leq \frac{1}{\sqrt{2}}$,
- $\theta' - \theta = 0,\ \pi,\ \arccos(-\frac{1}{2\gamma^2})$ *and* $-\arccos(-\frac{1}{2\gamma^2})$ *if* $\gamma \geq \frac{1}{\sqrt{2}}$.

They correspond to the values

- $\|y_\theta - y_{\theta'}\| = 0$ *if* $\theta' - \theta = 0$,
- $\|y_\theta - y_{\theta'}\| = 2$ *if* $\theta' - \theta = \pi$,
- $\|y_\theta - y_{\theta'}\| = \sqrt{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}$ *if* $\theta' - \theta = \pm\arccos(-\frac{1}{2\gamma^2})$.

Moreover, we have $\sqrt{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}} \geq 2$ *when* $\gamma \geq \frac{1}{\sqrt{2}}$.

Proof. Let $\theta, \theta' \in [0, 2\pi)$. One computes that

$$\|y_\theta - y_{\theta'}\|^2 = 4\sin^2\left(\frac{\theta - \theta'}{2}\right) + 2\gamma^2 \sin^2(\theta - \theta').$$

Consider the map $f \colon x \in [0, 2\pi) \mapsto 4\sin^2\left(\frac{x}{2}\right) + 2\gamma^2\sin^2(x)$. Its derivative is

$$\begin{aligned} f'(x) &= 4\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) + 4\gamma^2\cos(x)\sin(x) \\ &= 2\sin(x)\left(1 + 2\gamma^2\cos(x)\right). \end{aligned}$$

It vanishes when $x = 0$, $x = \pi$, or $x = \pm\arccos(-\frac{1}{2\gamma^2})$ if $\gamma \geq \frac{1}{\sqrt{2}}$. A computation shows that $f(0) = 0$, $f(\pi) = 4$ and $f\left(\pm\arccos\left(-\frac{1}{2\gamma^2}\right)\right) = 2\left(1 + \gamma^2 + \frac{1}{4\gamma^2}\right)$. $\square$

Lemma B.3. *For every $x \in E$ such that $x \neq 0$, the map $\theta \mapsto \|x - y_\theta\|$ admits at most two local maxima and two local minima.*

Proof. Consider the map $g \colon \theta \in [0, 2\pi) \mapsto \|x - y_\theta\|^2$. It can be written as

$$\begin{aligned} g(\theta) &= \|x\|^2 + \|y_\theta\|^2 - 2\langle x, y_\theta\rangle \\ &= \|x\|^2 + 1 + \frac{\gamma^2}{2} - 2\langle x, y_\theta\rangle. \end{aligned}$$

Let us show that its derivative g' vanishes at most four times on $[0, 2\pi)$, which would show the result. Using the expression of y_θ, we see that g' can be written as

$$g'(\theta) = a\cos(\theta) + b\sin(\theta) + c\cos(2\theta) + d\sin(2\theta),$$

where $a, b, c, d \in \mathbb{R}$ are not all zero. Denoting $\omega = \cos(\theta)$ and $\xi = \sin(\theta)$, we have $\xi^2 = 1 - \omega^2$, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\omega^2 - 1$ and $\sin(2\theta) = 2\cos(\theta)\sin(\theta) = 2\omega\xi$. Hence

$$g'(\theta) = a\omega + b\xi + 2c\omega^2 + 2d\omega\xi.$$

Now, if $g'(\theta) = 0$, we get

$$a\omega + 2c\omega^2 = -(b + 2d\omega)\xi \tag{16}$$

Squaring this equality yields $\left(a\omega + 2c\omega^2\right)^2 = (b + 2d\omega)^2(1 - \omega^2)$. This degree four equation, with variable ω, admits at most four roots. To each of these w, there exists a unique $\xi = \pm\sqrt{1 - w^2}$ that satisfies Equation 16. In other words, the corresponding $\theta \in [0, 2\pi)$ such that $\omega = \cos(\theta)$ is unique. We deduce the result. $\square$

Before studying the Čech filtration of Y, let us describe some geometric quantities associated to it. Using a symbolic computation software, we see that the curvature of Y is contant and equal to

$$\rho = \frac{\sqrt{1+8\gamma^2}}{1+2\gamma^2}.$$

In particular, we have $\rho \geq 1$ if $\gamma \leq 1$, and $\rho < 1$ if $\gamma > 1$. We also have an expression for the diameter of Y:

$$\frac{1}{2}\mathrm{diam}\,(Y) = \begin{cases} 1 & \text{if } \gamma \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}}\sqrt{1+\gamma^2+\frac{1}{4\gamma^2}} & \text{if } \gamma \geq \frac{1}{\sqrt{2}}. \end{cases}$$

It is a consequence of Lemma B.2. We now describe the reach of Y:

$$\mathrm{reach}(Y) = \begin{cases} \frac{1+2\gamma^2}{\sqrt{1+8\gamma^2}} & \text{if } \gamma \leq 1, \\ 1 & \text{if } \gamma \geq 1. \end{cases} \tag{17}$$

Let us prove this by using [AKC$^+$19, Theorem 3.4]. We define a bottleneck of Y as pair of distinct points $(y,y') \in Y^2$ such that the open ball $\mathcal{B}\left(\frac{1}{2}(y+y'), \frac{1}{2}\|y-y'\|\right)$ does not intersect Y. Its length is defined as $\frac{1}{2}\|y-y'\|$. Then the reach of Y is equal to

$$\mathrm{reach}(Y) = \min\left\{\frac{1}{\rho}, \delta\right\},$$

where $\frac{1}{\rho}$ is the inverse curvature of Y, and δ is the minimal length of bottlenecks of Y. As we computed, $\frac{1}{\rho}$ is equal to $\frac{1+2\gamma^2}{\sqrt{1+8\gamma^2}}$. Besides, according to Lemma B.2, a bottleneck $(y_\theta, y_{\theta'})$ has to satisfy $\theta' - \theta = \pi$ or $\pm\arccos(-\frac{1}{2\gamma^2})$. The smallest length is attained when $\theta' - \theta = \pi$, for which $\frac{1}{2}\|y_\theta - y_{\theta'}\| = 1$. It is straightforward to verify that the pair $(y_\theta, y_{\theta'})$ is indeed a bottleneck. Therefore we have $\delta = 1$, and we deduce the expression of $\mathrm{reach}(Y)$.

Last, the weak feature size of Y does not depend on γ and is equal to 1:

$$\mathrm{wfs}\,(Y) = 1. \tag{18}$$

We will prove it by using the characterization of Subsection 1.3: $\mathrm{wfs}\,(Y)$ is the infimum of distances $\mathrm{dist}\,(x,Y)$, where $x \in E$ is a critical point of the distance function d_Y. In this context, x is a critical point if it lies in the convex hull of its projections on Y. Remark that, if $x \neq 0$, then x admits at most two projections on Y. This follows from Lemma B.3. As a consequence, if x is a critical point, then there exists $y, y' \in Y$ such that x lies in the middle of the segment $[y,y']$, and the open ball $\mathcal{B}\,(x, \mathrm{dist}\,(x,Y))$ does not intersect Y. Therefore y' is a critical point of $y' \mapsto \|y-y'\|$, hence Lemma B.2 gives that $\|y-y'\| \geq 2$. We deduce the result.

We now describe the thickenings Y^t. They present four different behaviours:

- $0 \leq t < 1$: Y^t is homotopy equivalent to a circle,
- $1 \leq t < \frac{1}{2}\mathrm{diam}\,(Y)$: Y^t is homotopy equivalent to a circle,
- $\frac{1}{2}\mathrm{diam}\,(Y) \leq t < \sqrt{1+\frac{\gamma^2}{2}}$: Y^t is homotopy equivalent to a 3-sphere,
- $t \geq \sqrt{1+\frac{\gamma^2}{2}}$: Y^t is homotopy equivalent to a point.

Recall that, in the case where $\gamma \leq \frac{1}{\sqrt{2}}$, we have $\frac{1}{2}\mathrm{diam}\,(Y) = 1$. Consequently, the interval $\left[1, \frac{1}{2}\mathrm{diam}\,(Y)\right)$ is empty, and the second point does not appear in this case.

Study of the case $0 \le t < 1$. For $t \in [0,1)$, let us show that Y^t deform retracts on Y. According to Equation 18, we have $\mathrm{wfs}\,(Y) = 1$. Moreover, Equation 17 gives that $\mathrm{reach}(Y) > 0$. Using the results of Subsection 1.3, we deduce that Y^t is isotopic to Y.

Study of the case $1 \le t < \frac{1}{2}\mathrm{diam}\,(Y)$. Denote $z_\theta = \left(0, 0, \frac{\gamma}{\sqrt{2}}\cos(2\theta), \frac{\gamma}{\sqrt{2}}\sin(2\theta)\right)$, and define the circle $Z = \{z_\theta, \theta \in [0,\pi)\}$.

Figure 30: Representation of the set Y (black) and the circle Z (red).

We claim that Y^t deform retracts on Z. To prove so, we will define a continuous application $f \colon Y^t \to Z$ such that, for every $y \in Y^t$, the segment $[y, f(y)]$ is included in Y^t. This would lead to a deformation retraction of Y^t onto Z, via

$$(s, y) \in [0,1] \times Y^t \mapsto (1-s)y + sf(y).$$

Equivalently, we will define an application $\Theta \colon Y^t \to [0,\pi)$ such that the segment $[y, z_{\Theta(y)}]$ is included in Y^t.

Let $y \in Y^t$. According to Lemma B.3, y admits at most two projection on Y. We start with the case where y admits only one projection, namely y_θ with $\theta \in [0, 2\pi)$. Let $\overline{\theta} \in [0,\pi)$ be the reduction of θ modulo π, and consider the point $z_{\overline{\theta}}$ of Z. A computation shows that the distance $\|y_\theta - z_{\overline{\theta}}\|$ is equal to 1. Besides, since $y \in Y^t$, the distance $\|y_\theta - y\|$ is at most t. By convexity, the segment $\left[y, z_{\overline{\theta}}\right]$ is included in the ball $\overline{\mathcal{B}}\,(y_\theta, t)$, which is a subset of Y^t. We then define $\Theta(y) = \overline{\theta}$.

Now suppose that y admits exactly two projection y_θ and $y_{\theta'}$. According to Lemma B.2, these angles must satisfy $\theta' - \theta = \pi$. Indeed, the case $\|y_\theta - y_{\theta'}\| = \sqrt{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}$ does not occur since we chose $t < \frac{1}{2}\mathrm{diam}\,(Y) = \frac{\sqrt{2}}{2}\sqrt{1 + \gamma^2 + \frac{1}{4\gamma^2}}$. The angles θ and θ' correspond to the same reduction modulo π, denoted $\overline{\theta}$, and we also define $\Theta(y) = \overline{\theta}$.

Study of the case $t \in \left[\frac{1}{2}\mathrm{diam}\,(X), \sqrt{1 + \frac{\gamma^2}{2}}\right)$. Let $\mathbb{S}_3$ denotes the unit sphere of E. For every $v = (v_1, v_2, v_3, v_4) \in \mathbb{S}_3$, we will denote by $\langle v \rangle$ the linear subspace spanned by v, and by $\langle v \rangle_+$ the cone $\{\lambda v, \lambda \ge 0\}$. Moreover, we define the quantity

$$\delta(v) = \min_{y \in Y} \mathrm{dist}\,(y, \langle v \rangle_+)\,.$$

and the set

$$S = \{\delta(v)v, v \in \mathbb{S}_3\}\,.$$

We claim that S is a subset of Y^t, and that Y^t deform retracts on it. This follows from the two following facts: for every $v \in \mathbb{S}_3$,

1. $\delta(v)$ is not greater than $\frac{1}{2}\mathrm{diam}\,(Y)$,
2. $\langle v \rangle_+ \cap Y^t$ consists of one connected component: an interval centered on $\delta(v)v$, that does not contains the point 0.

Suppose that these assertions are true. Then one defines a deformation retraction of Y^t on S by retracting each fiber $\langle v\rangle_+ \cap Y^t$ linearly on the singleton $\{\delta(v)v\}$. We will now prove the two items.

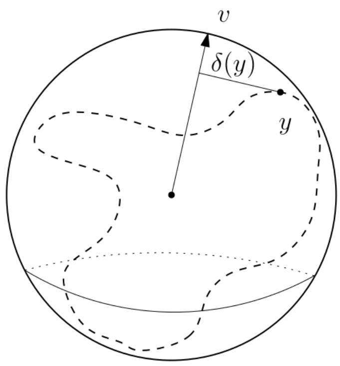


Figure 31: Representation of the set Y (dashed), lying on a 3-sphere of radius $\sqrt{1+\frac{\gamma^2}{2}}$.

Item 1.
Note that Item 1 can be reformulated as follows:

$$\max_{v\in\mathbb{S}_3}\min_{y\in Y} \operatorname{dist}(y,\langle v\rangle_+) \le \frac{1}{2}\operatorname{diam}(Y)\,. \tag{19}$$

Let us justify that the pairs (v, y) that attain this maximum-minimum are the same as in

$$\max_{v\in\mathbb{S}_3}\min_{y\in Y} \|y-v\|. \tag{20}$$

From the definition of $Y = \{y_\theta, \theta \in [0, 2\pi)\}$, we see that $\min_{y\in Y}\operatorname{dist}(y,\langle v\rangle_+) = \min_{y\in Y}\operatorname{dist}(y,\langle v\rangle)$. A vector $v \in \mathbb{S}_3$ being fixed, let us show that $y \mapsto \operatorname{dist}(y,\langle v\rangle)$ is minimized when $y \mapsto \|v-y\|$ is. Let $y \in Y$. Since v is a unit vector, the projection of y on $\langle v\rangle$ can be written as $\langle y, v\rangle\, v$. Hence $\operatorname{dist}(y,\langle v\rangle)^2 = \|\langle y, v\rangle\, v - y\|^2$, and expanding this norm yields

$$\operatorname{dist}(y,\langle v\rangle)^2 = \|y\|^2 - \langle y, v\rangle^2\,.$$

Expanding the norm $\|y-v\|^2$ and using that $\|y\|^2 = 1+\frac{\gamma^2}{2}$, we get $\langle y, v\rangle = \frac{1}{2}\left(2+\frac{\gamma^2}{2} - \|y-v\|^2\right)$. We inject this relation in the preceding equation to obtain

$$\operatorname{dist}(y,\langle v\rangle)^2 = -\left(\frac{\gamma}{2}\right)^4 + \gamma^2 + \frac{1}{4}\|y-v\|^2\left(4+\gamma^2-\|y-v\|^2\right).$$

Now we can deduce that $y \mapsto \operatorname{dist}(y,\langle v\rangle)^2$ is minimized when $y \mapsto \|y-v\|$ is minimized. Indeed, the map $\|y-v\| \mapsto \frac{1}{4}\|y-v\|^2\left(4+\gamma^2-\|y-v\|^2\right)$ is increasing on $\left[0,\frac{1}{2}(4+\gamma^2)\right]$. But $\|y-v\| \le \|y\|+\|v\| = \frac{1}{2}(4+\gamma^2)$.

We deduce that that studying the left hand term of Equation 19 is equivalent to studying Equation 20. We will denote by $g\colon \mathbb{S}_3 \to \mathbb{R}$ the map

$$g(v) = \min_{y\in Y}\|y-v\|. \tag{21}$$

Let $v \in \mathbb{S}_3$ that attains the maximum of g, and let y be a corresponding point that attains the minimum of $\|y-v\|$. The points v and y attains the quantity in Equation 19. In order to prove that $\operatorname{dist}(y,\langle v\rangle) \le \frac{1}{2}\operatorname{diam}(Y)$, let $p(y)$ denotes the projection of y on $\langle v\rangle$. We will show that there exists another point $y' \in Y$ such that $p(y)$ is equal to $\frac{1}{2}(y+y')$ Consequently, we would have $\|y-p(y)\| = \frac{1}{2}\|y'-y\| \le \frac{1}{2}\operatorname{diam}(Y)$, i.e.

$$\operatorname{dist}(y,\langle v\rangle) \le \frac{1}{2}\operatorname{diam}(Y)\,.$$

Remark the following fact: if $w \in \mathbb{S}_3$ is a unit vector such that $\langle p(y) - y, w \rangle > 0$, then for $\epsilon > 0$ small enough, we have

$$\operatorname{dist}(y, \langle v + \epsilon w \rangle) > \operatorname{dist}(y, \langle v \rangle).$$

Equivalently, this statement reformulates as $0 \leq \left\langle y, \frac{1}{\|v+\epsilon w\|}(v + \epsilon w) \right\rangle < \langle y, v \rangle$. Let us show that

$$\left\langle y, \frac{1}{\|v + \epsilon w\|}(v + \epsilon w) \right\rangle = \langle y, v \rangle - \epsilon\kappa + o(\epsilon), \tag{22}$$

where $\kappa = \langle p(y) - y, w \rangle > 0$, and where $o(\epsilon)$ is the little-o notation. Note that $\frac{1}{\|v+\epsilon w\|} = 1 - \epsilon \langle v, w \rangle + o(\epsilon)$. We also have

$$\frac{1}{\|v + \epsilon w\|}(v + \epsilon w) = v + \epsilon \left(w - \langle v, w \rangle v \right) + o(\epsilon).$$

Expanding the inner product in Equation 22 gives

$$\begin{aligned}
\left\langle y, \frac{1}{\|v + \epsilon w\|}(v + \epsilon w) \right\rangle &= \langle y, v \rangle + \epsilon \left(\langle y, w \rangle - \epsilon \langle v, w \rangle \langle y, v \rangle \right) + o(\epsilon) \\
&= \langle y, v \rangle + \epsilon \langle y - \langle y, v \rangle v, w \rangle + o(\epsilon) \\
&= \langle y, v \rangle + \epsilon \langle y - p(y), w \rangle + o(\epsilon),
\end{aligned}$$

and we obtain the result.

Next, let us prove that y is not the only point of Y that attains the minimum in Equation 21. Suppose that it is the case by contradiction. Let $w \in \mathbb{S}_3$ be a unit vector such that $\langle p(y) - y, w \rangle > 0$. For ϵ small enough, let us prove that the vector $v' = \frac{1}{\|v+\epsilon w\|}(v + \epsilon w)$ of $\mathbb{S}_3$ contradicts the maximality of v. That is, let us prove that $g(v') > g(v)$. Let $y' \in Y$ be a minimizer $\|y' - v'\|$. We have to show that $\|y' - v'\| > \|y - v\|$. This would lead to $g(v') > g(v)$, hence the contradiction.

Expanding the norm yields

$$\|v' - y'\|^2 = \|v' - v + v - y'\|^2 \geq \|v' - v\|^2 + \|v - y'\|^2 - 2 \langle v' - v, v - y' \rangle.$$

Using $\|v' - v\|^2 \geq 0$ and $\|v - y'\|^2 \geq \|v - y\|^2$ by definition of y, we obtain

$$\|v' - y'\|^2 \geq \|v - y\|^2 - 2 \langle v' - v, v - y' \rangle.$$

We have to show that $\langle v' - v, y - y' \rangle$ is positive for ϵ small enough. By writing $v - y' = v - y + (y - y')$ we get

$$\langle v' - v, v - y' \rangle = \langle v' - v, v \rangle - \langle v' - v, y \rangle + \langle v' - v, y - y' \rangle$$

According to Equation 22, $-\langle v' - v, y \rangle = \epsilon\kappa + o(\epsilon)$. Besides, using $v' - v = \epsilon(w - \langle v, w \rangle v) + o(\epsilon)$, we get $\langle v' - v, v \rangle = o(\epsilon)$. Last, Cauchy-Schwarz inequality gives $|\langle v' - v, y - y' \rangle| \leq \|v' - v\| \|y - y'\|$. Therefore, $\langle v' - v, y - y' \rangle = O(\epsilon)\|y - y'\|$, where $O(\epsilon)$ is the big-o notation. Gathering these three equalities, we obtain

$$\langle v' - v, v - y' \rangle = o(\epsilon) + \epsilon\kappa + O(\epsilon)\|y - y'\|.$$

As we can read from this equation, if $\|y - y'\|$ goes to zero as ϵ does, then $\langle v' - v, v - y' \rangle$ is positive for ϵ small enough. Observe that v' goes to v when ϵ goes to 0. By assumption y is the only minimizer in Equation 21. By continuity of g, we deduce that y' goes to y.

By contradiction, we deduce that there exists another point y' which attains the minimum in $g(v)$. Note that it is the only other one, according to Lemma B.3. Let us show that $p(y)$ lies in the middle of the segment $[y, y']$. Suppose that it is not the case. Then $p(y) - y$ is not equal to $-(p(y') - y')$, where $p(y')$ denotes the projection of y' on $\langle v \rangle$. Consequently, the half-spaces $\{w \in E, \langle p(y) - y, w \rangle > 0\}$ and $\{w \in E, \langle p(y') - y', w \rangle > 0\}$ intersects. Let w be any vector in the intersection. For $\epsilon > 0$, denote $v' = \frac{1}{\|v+\epsilon w\|}(1 + \epsilon w)$. If ϵ is small enough, the same reasoning as before shows that v' contradicts the maximality of v.

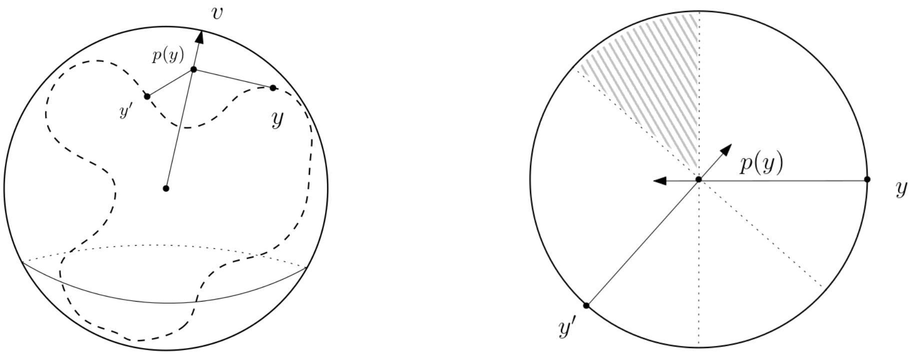


Figure 32: Left: Representation of the situation where y and y' are minimizers of Equation 21. Right: Representation in the plane passing through the points y, y' and $p(y)$. The dashed area corresponds to the intersection of the half-spaces $\{w \in E, \langle p(y) - y, w \rangle > 0\}$ and $\{w \in E, \langle p(y') - y', w \rangle > 0\}$.

Item 2.
Let $v \in \mathbb{S}_3$. The set $\langle v \rangle_+ \cap Y^t$ can be described as

$$\langle v \rangle_+ \cap \bigcup_{y \in Y} \overline{\mathcal{B}}(y, t) .$$

Let $y \in Y$ such that $\langle v \rangle_+ \cap \overline{\mathcal{B}}(y, t) \neq \emptyset$. Denote by $p(y)$ the projection of y on $\langle v \rangle_+$. It is equal to $\langle y, v \rangle v$. Using Pythagoras' theorem, we obtain that the set $\langle v \rangle_+ \cap \overline{\mathcal{B}}(y, t)$ is equal to the interval

$$\left[p(y) \pm \sqrt{t^2 - \operatorname{dist}(y, \langle v \rangle)^2} v \right].$$

Using the identity $\operatorname{dist}(y, \langle v \rangle)^2 = \|y\| - \langle y, v \rangle^2 = 1 + \frac{\gamma^2}{2} - \langle y, v \rangle^2$, we can write this interval as

$$\left[I_1(y) \cdot v, \; I_2(y) \cdot v \right],$$

where $I_1(y) = \langle y, v \rangle - \sqrt{\langle y, v \rangle^2 - (1 + \frac{\gamma^2}{2} - t^2)}$ and $I_2(y) = \langle y, v \rangle + \sqrt{\langle y, v \rangle^2 - (1 + \frac{\gamma^2}{2} - t^2)}$. Seen as functions of $\langle y, v \rangle$, the map I_1 is decreasing, and the map I_2 is increasing (see Figure 33). Let $y^* \in Y$ that minimizes $\operatorname{dist}(y, \langle v \rangle)$. Equivalently, y^* maximizes $\langle y, v \rangle$. It follows that the corresponding interval $\left[I_1(y^*) \cdot v, \; I_2(y^*) \cdot v \right]$ contains all the others. We deduce that the set $\langle v \rangle_+ \cap Y^t$ is equal to this interval.

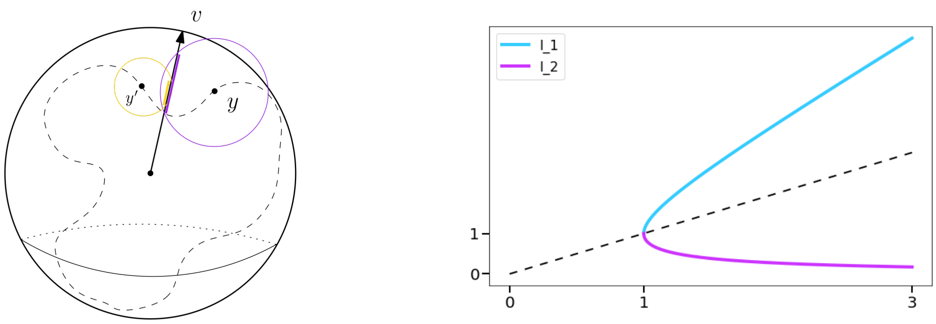


Figure 33: Left: Representation of two intervals $\langle v \rangle_+ \cup \overline{\mathcal{B}}(y, t)$ and $\langle v \rangle_+ \cup \overline{\mathcal{B}}(y', t)$. Right: Representation of the maps $x \mapsto x \pm \sqrt{x^2 - 1}$.

Study of the case $t \geq \sqrt{1+\frac{1}{2}\gamma^2}$. For every $y \in Y$, we have $\|y\| = \sqrt{1+\frac{1}{2}\gamma^2}$. Therefore, if $t \geq \sqrt{1+\frac{1}{2}\gamma^2}$, then Y^t is star shaped around the point 0, hence it deform retracts on it.

Čech bundle filtration of Y. To close this subsection, let us study the Čech bundle filtration $(\mathbb{Y}, \mathbb{p})$ of Y. According to Equation 5, its filtration maximal value is $t^{\max}(Y) = t^{\max}_{\gamma}(X) = \frac{\gamma}{\sqrt{2}}$. Note that $\frac{\gamma}{\sqrt{2}}$ is lower than $\frac{1}{2}\mathrm{diam}(Y)$. Consequently, only two cases are to be studied: $t \in [0,1)$, and $t \in \left[1, \frac{1}{2}\mathrm{diam}(Y)\right)$.

The same argument as in Subsection B.2 yields that for every $t \in [0,1)$, the persistent Stiefel-Whitney class $w_1^t(Y)$ is equal to $w_1^0(Y)$. Accordingly, for every $t \in \left[1, \frac{1}{2}\mathrm{diam}(Y)\right)$, the class $w_1^t(Y)$ is equal to $w_1^1(Y)$. Let us show that $w_1^0(Y)$ is zero, and that $w_1^1(Y)$ is not.

First, remark that the map $p^0 \colon Y \to \mathcal{G}_1(\mathbb{R}^2)$ can be seen as the normal bundle of the circle. Hence $(p^0)^* \colon H^*(Y) \leftarrow H^*(\mathcal{G}_1(\mathbb{R}^2))$ is nontrivial, and we deduce that $w_1^0(Y) = 0$. As a consequence, the persistent Stiefel-Whitney class $w_1^t(X)$ is nonzero for every $t < 1$.

Next, consider $p^1 \colon Y^1 \to \mathcal{G}_1(\mathbb{R}^2)$. Recall that Y^1 deform retracts on the circle

$$Z = \left\{ \left(0, 0, \frac{\gamma}{\sqrt{2}}\cos(2\theta), \frac{\gamma}{\sqrt{2}}\sin(2\theta)\right), \theta \in [0,\pi) \right\}.$$

Seen in $\mathbb{R}^2 \times \mathcal{M}(\mathbb{R}^2)$, we have

$$Z = \left\{ \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \gamma \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{pmatrix} \right), \theta \in [0,\pi) \right\}.$$

Notice that the map $q \colon Z \to \mathcal{G}_1(\mathbb{R}^2)$, the projection on $\mathcal{G}_1(\mathbb{R}^2)$, is injective. Seen as a map between two circles, it has degree (modulo 2) equal to 1. We deduce that $q^* \colon H^*(Z) \leftarrow H^*(\mathcal{G}_1(\mathbb{R}^2))$ is nontrivial. Now, remark that the map q factorizes through p^1:

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr, "q"'] & & Y^1 \arrow[dl, "p^1"] \\ & \mathcal{G}_1(\mathbb{R}^2) & \end{tikzcd}$$

It induces the following diagram in cohomology:

$$\begin{tikzcd} H^*(Z) & & H^*(Y^1) \arrow[ll, "\sim"'] \\ & H^*(\mathcal{G}_1(\mathbb{R}^2)) \arrow[ul, "q^*"] \arrow[ur, "(p^1)^*"'] & \end{tikzcd}$$

Since q^* is nontrivial, this commutative diagram yields that the persistent Stiefel-Whitney class $w_1^1(Y)$ is nonzero. As a consequence, the persistent Stiefel-Whitney class $w_1^t(Y)$ is nonzero for every $t \geq 1$.

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